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Spline identification of linear time varying and a class of nonlinear systems.

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SPLINE IDENTIFICATION OF LINEAR TIME VARYING AND A CLASS OF NONLINEAR SYSTEMS

by

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ABSTRACT

Modern control theory, in its application to control and optimization, requires the development of an appropriate mathematical model to describe the dynamic behaviour of the system throughout its operating range. Numerous techniques have been developed in recent years on this subject. In this work the use of cubic splines is considered for identification of linear nonstationary systems and a class of nonlinear systems.

The use of cubic splines in parameter identification initially proposed by Bellman in 1971, was extended by Balatoni (1973) for linear lumped and distributed constant parameter systems by the use of on-line recursive estimation. In the present work it was desired to develop a simple algorithm for on-line identification of linear time varying and nonlinear systems by the application of cubic splines.

First, the on-line identification scheme for constant parameter linear systems is extended to time varying systems by assuming the parameters to be either piecewise constant or varying randomly. No apriori knowledge about parameter variation is assumed in this procedure. The procedure is compared with central difference method and is shown to be superior, especially when the available data is noisy.

Secondly, the use of cubic splines in the identification of a class of nonlinear systems is considered. An on-line procedure is developed for the case of unknown end points. The procedure involves on-line derivative estimation by cubic splines which results in a system of linear algebraic equations that may be solved by recursive least squares estimation procedure. The spline method, in conjunction with state-

variable approach, is also used for nonlinear multidimensional systems described by a set of first order differential equations. Due to the minimum norm property of cubic spline, the estimated derivatives are shown to be smooth, an important feature when the available data is subject to random noise. A comparison with central difference method reveals the spline technique to be more accurate and rapidly converging.

In the initial stage of the work, the system output was obtained by numerical solution of system equations in digital computer. In the later stage of the work, the system output was obtained by simulating the system on an EAI 580 Analog/Hybrid Computer.

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CHAPTER I

INTRODUCTION

1.1. Purpose of Identification

In control problems the final goal is the design of optimum control strategies for a particular system. Modern techniques for analysis and design of systems require a mathematical model of the system under study. In preliminary studies, the system equations and parameter values may be obtained from theoretical studies, manufacturer's specifications, etc. However, it is well known that the knowledge about a system and its environment, which is required to design a controller, is seldom available apriori. Even if the equations governing a system are known, in principle, it happens that knowledge of a particular parameter is missing. It is possible to perform experiments on the system in order to obtain the lacking knowledge. When hardware is available, tests may be made to determine either the appropriate equation or the parameter values or both. The models used in control theory, with very few exceptions, are parametric models described by state equations. The desire to determine such models from experimental data has naturally renewed the interest of control engineers in parameter estimation and related techniques.

1.2. Different Identification Techniques

Various test techniques are available, such as frequency response, impulse and step testing. These techniques undoubtedly got very much popularity in the classical control theory because the techniques made it possible to accurately determine the transfer functions, which are needed to apply the synthesis method. But these methods are not always applicable and the accuracy of the resulting model depends upon the analyst who processes the data. Moreover, the preceding techniques need very accurate data. As a result, identification schemes based on stochastic models are needed.

A number of survey papers on system identification have been written, based on many references, by Eykhoff, Grinten, Kawkernaak and Veltman [12], Cuenod and Sage [9], Eykoff [13], Balakrishnan and Paterka [5], and Astrom and Eykhoff [2]. These authors divided different available identification techniques into groups such as, least square/general least squares, maximum likelihood, Bayes estimation, use of deterministic model, etc., and attempted to discuss each group in brief. However, most of their discussion was restricted to constant parameter linear model. A few of them also discussed non-linear system identification. Among the numerous identification techniques, some of the important contributions are due to Kalman [16] who viewed the identification problem as an optimal linear filtering problem. Many authors have applied the techniques using cross-correlation with different degree of success [19,22]. However, our interest in this work was restricted to on-line identification of linear time-varying and non-linear systems.

The most accurate techniques for identification of time-varying systems are, in general, off-line schemes, such as maximum likelihood estimation [3,4,8]. Among the on-line techniques, Paterka and Smuk [25], using Bayesian approach, have shown that it is particularly suitable for parameters varying randomly. Larminant and Tallec [10] have developed an identification technique that may be viewed as a non-linear filtering problem, approximately optimal. It is basically a Kalman Bucy filter, where the second order terms of Taylor's series expansion of non-linear functions are taken into account. It is also concerned with variance estimation problem, for its knowledge allows the implementation of the filter.

Park and Shen [23] also presented an identification algorithm for a class of nonstationary processes, an on-line procedure for estimating the unknown time varying stochastic parameter associated with a discrete linear dynamic system. The algorithm is based on analysis of bias in a stochastic steepest descent approach and subsequent application of Dupac's [11] dynamic stochastic approximation method. The upper bound of the mean squared error was minimized at each instant by choosing the scalar gain sequence to accelerate the convergence. Comparing with the extended Kalman filter, the authors have shown that their algorithm requires little apriori statistical information on plant input disturbances; also it is less sensitive to initial guess. However, in their technique, the variation of the non-stationary parameter must be known apriori.

There are a variety of algorithms for non-linear system identification, with a number of survey papers in the subject [9; 5; 33]. Most of these techniques are based on converting the identification problem to an approximation problem by postulating a structure. The few non-parametric techniques available are computationally extremely time consuming. One popular approach to the non-linear ~~estima~~tion is to linearize the system about the working range and apply the optimal linear sequential filter [15,17].

An on-line estimation of non-linear process parameter has been developed by Jackson and Rippin [21], upon a gradient method of minimizing the sum of squared errors. The gradient vector was multiplied by a modifying matrix to give a parameter adjustment vector, a suitable fraction of which was used to update the model parameter. Eventually, it is an exact correspondence with linear sequential least squares filter.

Gardiner [14] developed a frequency domain technique for identification of non-linear systems with single valued non-linearity which allows the simple cascade form representation of the system. An identification scheme for non-linear multivariable systems by pseudo-sensitivity functions, from the point of view of quasilinearization has been developed by Matusek and Milovanovic [20], Sutek and Varga [32] have described a procedure that utilizes discontinuous orthogonal filters and multivariable pseudo-random signals.

A multitude of techniques and various paper references can be obtained in the survey papers mentioned above.

1.3. Problem Statement

The aim of this work was to develop an on-line identification scheme for linear time varying systems and a class of non-linear systems by approximating the output functions with cubic splines. Polynomial spline functions were developed primarily for function interpolation. Among the various orders of polynomial splines, cubic spline has a number of attractive properties. Since derivative estimation needed for the identification scheme is sensitive to the accuracy of data, cubic splines got much attention, for the resulting derivatives are smooth, a property attributable partly to the best approximation characteristics, partly to the minimum curvature property.

In 1971, Bellman and Roth [7] suggested the use of splines in the identification of lumped parameter systems. Philipson [26] in the same year suggested the use of splines in distributed system identification. After 1971, Shridhar and Balatoni [29, 30, 31] started working on constant parameter lumped and distributed linear system identification with the application of cubic spline by on-line recursive relationship. The authors have shown that spline identification has better convergence properties and results in more accurate parameter estimates, compared to other finite difference methods especially when the available data is noisy.

In fact, the present work is an extension of the work of Balatoni (1973) to on-line identification of linear time varying and non-linear systems. The form of equation describing the system is assumed to be known. No information about the parameter of its variation was assumed to be known apriori. The application of cubic spline is considered for single input-output as well as multiple-input-output systems.

The class of non-linear systems considered here, are linear-in-parameter.

1.4. Organization

In chapter 2, the basic definitions and the important properties of cubic splines are described.

In chapter 3, an identification algorithm for lumped stationary process developed by Balatoni [6], is discussed.

In chapter 4, the use of cubic splines has been extended to recursive on-line identification of a linear time varying system with or without prior knowledge of parameter variation. On-line recursive identification for non-linear system is also developed. Lastly, identification of systems described by a set of first order differential equation has been considered. It has been shown by variance estimation that identification of cubic splines is superior to the central difference method.

In chapter 5, simulation results of the said techniques are presented. Computer aided results are discussed for linear nonstationary and non-linear system identification techniques developed in chapter 4. The effect of various mesh size has been also discussed. Lastly, this identification technique was applied to systems implemented on an EAI 580 Analog/Hybrid Computer.

CHAPTER 2

THE THEORY OF SPLINES

2.1. Introduction

One of the most direct ways to approximate a function defined at a finite set of points is to fit a polynomial of suitable degree. In addition to its simplicity, polynomial approximation has some advantageous properties. But the main drawback is that it becomes excessively oscillatory between the nodes as the number of points to be interpolated increases. This drawback can be eliminated by considering a set of piecewise polynomials, i.e., functions that are polynomial, possibly different polynomials in different subdomains of the domain on which we are approximating. These piecewise polynomial functions together with some continuity conditions on the approximating polynomials and their derivatives at the nodes are known as splines.

The spline approximation in its recent form was first studied by Schoenberg (1946). After 1946, in particular, Schoenberg and Whitney (1949-1953) first obtained criteria for the existence of certain spline interpolation.

There is a close relationship between spline theory and beam theory. Engineers and draftsmen have for a long time used thin rods, called splines, to fair curves through given points by attaching lead weights on different points, the spline was made to pass through the specified points. The 'strain energy' minimized by such splines is proportional, approximately, to the integral of the square of the second derivatives of the spline.

2.2. Mathematical Spline

The mathematical spline comes by replacing the rod spline with its elastica and approximating the latter by a piecewise polynomial. The mathematical cubic spline is continuous along with its first and second derivatives and with discontinuity (in general) in its third derivative, quite analogous to the draftman's spline having continuous curvature but jumps occuring at rate of curvature. In constrast, the draftsman did not use lead weights at the specified points through which the spline must pass.

An important property of the spline that makes it effective in interpolation is its striking convergence property, which may be explained as: if $f^{(q)}(x)$* is continuous on [a,b] (q=0,1,2,...,4) the spline approximation $S_\Delta(f,x)$ converges to $f(x)$ on a sequence of meshes on [a,b] at least as rapidly as the approach to zero of the q^{th} power of the mesh norm $||\Delta|| = \max h_j$**. Similarly, $S_\Delta^p(f,x)$ converges to $f^p(x)$ $(0 \le p < q)$ at least as rapidly as the (q-p)th power of the mesh norm approaches zero where the mesh Δ is defined as

$$\Delta: a = x_o < x_1 < \cdots < x_n = b.$$

The convergence rate is also claimed to be optimal [1].

2.3. Cubic Spline

Cubic spline $S_\Delta(x)$ is a piecewise continuous function on a mesh with the following properties:

(i) the cubic spline $S_\Delta(x)$ is cubic in each subinterval $x_{i-1} < x < x_i$, $i = 1, 2, \ldots, N$.

(ii) The cubic spline $S_\Delta(x)$ is continuous together with its first and second derivative. On [a,b] the second derivative is also piecewise linear.

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* $f^{(q)}(x) = d^q f(x) / dx^q$

** $h_j = x_j - x_{j-1}$

(iii) $S_\Delta(x)$ satisfies the equation $S_\Delta(x_i) = f(x_i)$, $i = 0, 1, \ldots, N$, where $f(x)$ is the function being approximated by $S_\Delta(x)$.

The close relation of cubic spline to the draftsman's spline (as a result of thin beam approximation) has lead to many of its important properties, which can be effectively applied to problems in numerical approximations. The spline proves to be an effective tool in the elementary process of interpolation and approximate integration. An outstanding characteristic is its effectiveness in numerical differentiation, especially, when the available data is only approximate.

Historically, the intrinsic properties of the spline were well hidden. After the introduction of the spline by Schoenberg, more than a decade elapsed before the first intrinsic property was uncovered. The property which we refer to as the minimum norm property was obtained by Holloday.

Theorem (Holladay): Let Δ: $a = x_o < x_1 < \cdots < x_n = b$ and a set of real numbers $\{y_i\}$ $(i = 0, 1, \ldots, N)$ be given, then of all function $f(x)$ having a continuous second derivative on $[a,b]$ and such that $f(x_i) = y_i$ $(i = 0, 1, \ldots, N)$, the spline function $S_\Delta(f,x)$ with junction points at the x_i and with $S''_\Delta(f,a) = S''_\Delta(f,b) = 0$ minimizes the integral

$$\int_a^b |f''(x)|^2 \, dx.$$

The integral is often a good approximation to the integral of the square of curvature for a curve $y = f(x)$, and hence the theorem may be called the minimum curvature property or minimum norm property in the sense of Ahlberg [1].

Theorem: Given a mesh Δ: $x_o < x_1 < \cdots < x_n = b$, then of all cubic splines $S_\Delta(x)$ with $S'_\Delta(a) = f'(a)$ and $S'_\Delta(b) = f'(b)$, the spline $S_\Delta(f;x)$ that interpolates the function $f(x)$ at the mesh points furnishes the best approximation;

i.e., if $S_\Delta(f;x)$ is the cubic spline interpolating to the function $f(x)$ and $S_\Delta(x)$ is any cubic spline with the end conditions as said above, the error functional $E = \int_a^b [f''(x) - S''_\Delta(x)]^2\, dx$ is minimized when $S''_\Delta(x) = S''_\Delta(f;x)$. This minimum is given by

$$\min(E) = \int_a^b [f''(x)]^2\, dx - \int_a^b [S''_\Delta(f;x)]^2\, dx \qquad 2.3.1$$

The quantity E is a measure of approximation of $S''_\Delta(x)$ to $f''(x)$ on [a,b]. This is known as the best approximation property of spline interpolation.

The proof of the above theorems can be found in the text [27].

2.4. Equation Defining Cubic Spline

The cubic spline $S_\Delta(x)$ interpolating to a function $f(x)$ in the interval [a,b] to the values y_i, may be defined in terms of its first and second derivatives. Define $h_j = x_j - x_{j-1}$, $S'_\Delta(x)$ as m_j and $S''_\Delta(x)$ as M_j, for nonperiodic spline.

The defining equation becomes

$$\begin{bmatrix} 2 & \mu_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda_1 & 2 & \mu_1 & & & 0 & 0 & 0 \\ 0 & \lambda_2 & 2 & \mu_2 & & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 2 & \mu_{N-2} & 0 \\ 0 & 0 & 0 & 0 & & \lambda_{N-1} & 2 & \mu_{N-1} \\ 0 & 0 & 0 & 0 & & 0 & \lambda_N & 2 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ \vdots \\ m_{N-2} \\ m_{N-1} \\ m_N \end{bmatrix} = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_{N-2} \\ C_{N-1} \\ C_N \end{bmatrix} \qquad 2.4.1$$

where $\lambda_j = \dfrac{h_{j+1}}{h_j + h_{j+1}}$, $\mu_j = 1 - \lambda_j$ $(j = 1, 2, \ldots, N-1)$ 2.4.2.

and $$C_j = 3\lambda_j \frac{y_j - y_{j-1}}{h_j} + 3\mu_j \frac{y_{j+1} - y_j}{h_{j+1}} \quad (j = 1,2,\ldots,N-1) \qquad 2.4.3$$

The values of λ_j, μ_j and C_j for $j = 0$ and N, depends upon the choice of end conditions. If the second derivative at the end points are known, then the following equations may be used

$$\mu_o = \lambda_N = 1 \qquad 2.4.4.$$

$$C_j = 3 \frac{y_1 - y_o}{h_1} - \frac{h_1}{2} y_o'' \qquad 2.4.5(a)$$

and $$C_N = 3 \frac{y_N - y_{N-1}}{h_N} + \frac{h_N}{2} y_N'' \qquad 2.4.5(b)$$

If the first derivative at the end points are known, then the equations to be used are,

$$\mu_o = \lambda_N = 0 \qquad 2.4.6$$

and $$C_j = 2y_j' \quad (j = 0, N). \qquad 2.4.7$$

The quantities M_j's can be found by the equation

$$M_j = \frac{2m_{j-1}}{h_j} + \frac{4m_j}{h_f} - 6 \frac{y_j - y_{j-1}}{h_j^2} \qquad 2.4.8$$

Similarly the spline defining equation with second derivatives becomes

$$\begin{bmatrix} 2 & \lambda_o & 0 & \cdots & 0 & 0 & 0 \\ \mu_1 & 2 & \lambda_1 & \cdots & 0 & 0 & 0 \\ 0 & \mu_2 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \\ 0 & 0 & 0 & \cdots & 2 & \lambda_{N-2} & 0 \\ 0 & 0 & 0 & \cdots & \mu_{N-1} & 2 & \lambda_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & \mu_N & 2 \end{bmatrix} \begin{bmatrix} M_o \\ M_1 \\ M_2 \\ \vdots \\ M_{N-2} \\ M_{N-1} \\ M_N \end{bmatrix} = \begin{bmatrix} d_o \\ d_1 \\ d_2 \\ \vdots \\ d_{N-2} \\ d_{N-1} \\ d_N \end{bmatrix} \qquad 2.4.9$$

where $d_j = 6 \dfrac{[(y_{j+1} - y_j) / h_{j+1}] - [(y_j - y_{j-1}) / h_j]}{h_j + h_{j+1}}$,

$[j = 1, 2, \ldots, N]$ 2.4.10

when y_o' and y_N' are known, the boundary values become

$$\lambda_o = \mu_N = 1 \qquad 2.4.11$$

$$d_o = \frac{6}{h_1}\left(\frac{y_1 - y_o}{h_1} - y_o'\right) \qquad 2.4.12(a)$$

and $$d_N = \frac{6}{h_N}\left(y_N' - \frac{y_N - y_{N-1}}{h_N}\right) \qquad 2.4.12(b)$$

when y_o'' and y_N'' are known, then

$$\lambda_o = \mu_N = 0 \qquad 2.4.13$$

and $d_j = 2M_j$, $[j = 0, N]$ 2.4.14

may be used.

The first derivative of the spline function becomes

$$m_j = \frac{h_j}{6} M_{j-1} + \frac{h_j}{3} M_j + \frac{y_j - y_{j-1}}{h_j} \qquad 2.4.15$$

2.5. End Conditions

The effect of end conditions, when chosen with discretion, dampens rapidly as one moves from extremities. However, it creates a problem, when one needs to know the derivative accurately at the end. If the slope of the end points are known, then conditions as given in section 2.4 are applicable as a justified end condition. However, in most cases, the slopes are not known. In such cases, conditions $M_o = M_N = 0$ can be used (these corresponds to placing simple supports at the beam ends). In the absence of other specifications $M_o = M_1$, $M_{N-1} = M_N$ is convenient. The resulting fit, however, may exhibit a large error on the values of M_j near the extremities when this particular end condition is in conflict. The more general end conditions are $M_o = \lambda_o M_1$, $M_N = \mu_N . M_{N-1}$, where λ_o and μ_N are adjusted so as to be consistent with the particular case.

2.6. Existence and Uniqueness

For any approximating or interpolating function, the question of existence and uniqueness is obvious. It has been shown [1], with application of Gershgorin's theorem, that periodic splines with prescribed ordinates at mesh points always exist and are unique and the same is true for nonperiodic splines having either cantilivered ends (m_o and m_N specified), or simple supports ($M_o = 0$, $M_N = 0$), or prescribed end moments, that is, simple supports at points beyond the end extremities (e.g., $M_o = \lambda M_1$, $M_N = \mu M_{N-1}$, $0 < \lambda < 1$, $0 < \mu < 1$).

The matrix formed by the coefficients of M_j or m_j as seen from equation (2.4.1) and (2.4.9) is tridiagonal and in general possesses the property of diagonal domance. When the main diagonal of the matrix is dominant, the inverse matrix exists and a bound can be found on the norm of the inverse matrix. For this matrix, the norm of the inverse is shown[6] to be ≤ 1. The properties of the coefficient matrix, namely diagonal dominance, existence of the inverse and bounded norm, merely add to the excellent convergence properties of cubic splines.

2.7. Equal Intervals

When the mesh Δ on $[a,b]$ are of equal length, the inverse of the coefficient matrix (2.4.1) or (2.4.9) takes a much simpler form as shown by [1]. The inversion process is given as follows.

Introduce the n x n determinant

$$D_n(\lambda) = \begin{vmatrix} 2 & \lambda & & & \\ 1-\lambda & 2 & \lambda & & \bigcirc \\ & & \cdots & & \\ & & 1-\lambda & 2 & \lambda \\ \bigcirc & & & 1-\lambda & 2 \end{vmatrix} \qquad 2.7.1$$

The determinant with $D_{-1}(\lambda) = 0$, $D_0(\lambda) = 1$ and $D_1(\lambda) = 2$ satisfies the equation

$$D_n(\lambda) - 2\,D_{n-1}(\lambda) + (1-\lambda)\,D_{n-2}(\lambda) = 0\,, \qquad 2.7.2$$

so that

$$D_n(\lambda) = \frac{[1 + (1-\lambda+\lambda^2)^{\frac{1}{2}}]^{n+1} - [1 - (1-\lambda+\lambda^2)^{\frac{1}{2}}]^{n+1}}{2(1-\lambda+\lambda^2)^{\frac{1}{2}}} \qquad 2.7.3$$

Setting $D_n = D_n(\frac{1}{2})$, $D_0 = 1$, $D_{-1} = 0$, we have

$$D_n = \frac{(1+3^{\frac{1}{2}})^{n+1} - (1-3^{\frac{1}{2}})^{n+1}}{3^{\frac{1}{2}}} \qquad 2.7.4$$

Now, define the n x n determinant $Q_n(\alpha)$ as

$$Q_n(\alpha) = \begin{vmatrix} 2 & \alpha & & & \bigcirc \\ \frac{1}{2} & 2 & \frac{1}{2} & & \\ & \frac{1}{2} & 2 & \frac{1}{2} & \\ & & \cdots & & \\ \bigcirc & & & \frac{1}{2} & 2 \end{vmatrix} \qquad 2.7.5$$

Here $Q_1(\alpha) = 2$. We define $Q_0(\alpha) = 1$. For $n \geq 1$

$$Q_n(\alpha) = 2D_{n-1} - (\alpha/2)\, D_{n-2}\,. \qquad 2.7.6$$

$Q_n(\alpha)$ also satisfies the same difference equation as D_n. The coefficient determinant of A becomes

$$|A| = 2Q_N(\mu_N) - \lambda_o\, Q_{N-1}(\mu_N) / 2 \qquad 2.7.7(a)$$

$$= 2Q_N(\lambda_o) - \mu_N\, Q_{N-1}(\lambda_o) / 2 \qquad 2.7.7(b)$$

The elements of the inverse matrix A^{-1} are obtained from the cofactors of the transpose matrix as

$$A_{ij}^{-1} = \frac{(-1)^{i+j}\, Q_i(\lambda_o)\, Q_{N-j}(\mu_N)}{2^{j-i}\, |A|} \quad (0 < i \leq j \leq N) \qquad 2.7.8(a)$$

$$A_{ij}^{-1} = \frac{(-1)^{i+j}\, Q_j(\lambda_o)\, Q_{N-i}(\mu_N)}{2^{i-j}\, |A|} \quad (0 \leq j \leq i < N) \qquad 2.7.8(b)$$

$$A_{0,j}^{-1} = \frac{(-1)^{j}\, \lambda_o\, Q_{N-j}(\mu_N)}{2^{j-1}\, |A|} \quad (0 < j \leq N) \qquad 2.7.8(c)$$

$$A_{N,j}^{-1} = \frac{(-1)^{N+j}\, \mu_N\, Q_j(\lambda_o)}{2^{N-j-1}\, |A|} \quad (0 \leq j < N) \qquad 2.7.8(d)$$

2.8. Approximate Differentiation and Integration

An important application of spline functions in numerical analysis is in the field of numerical integration and differentiation.

Solving m_j's (j = 0, 1, ..., N) in equation (2.4.1) and M_j's (j = 0, 1, ..., N) in equation (2.4.9) gives the first and second derivatives respectively. The resulting derivatives at the meshpoints are smooth due to the best approximation and minimum norm property. To get other derivatives, this procedure may be followed: First, compute the first derivative by interpolating ordinates with cubic spline, then set another spline through the first derivatives to get the second derivatives and repeat this procedure to get other derivatives. The procedure is known as the spline on spline procedure.

Integration may also be performed with the help of cubic splines, a direct integration of equation (2.4.9) gives the relation

$$\int_{x_{j-1}}^{x_j} S_\Delta(x)\, dx = \frac{f_{j-1} + f_j}{2} h_j - \frac{M_{j-1} + M_j}{24} h_j^3 \qquad 2.8.1$$

So that

$$\int_a^b S_\Delta(x)\, dx = \sum_{j=1}^{N} \frac{f_{j-1} + f_j}{2} h_j - \sum_{j=1}^{N} \frac{M_{j-1} + M_j}{24} h_j^3 \qquad 2.8.2$$

which in the case of equal intervals becomes

$$\int_a^b S_\Delta(x)\, dx = h \sum \frac{f_{j-1} + f_j}{2} - \frac{h^3}{12} \sum \frac{M_{j-1} + M_j}{2} \qquad 2.8.3$$

2.9. Higher Order Polynomial Splines

The attempt to extend the concept of cubic spline to higher order polynomial spline arises naturally. A detailed investigation of intrinsic properties of polynomial spline of odd degree is presented in [1]:

The most significant result in connection with polynomial spline is this: there is an essential difference between splines of odd and even degrees. It has been found that polynomial splines of odd degrees interpolating to a prescribed function at mesh points with specified end conditions always exists, but polynomial splines of even degrees interpolating to a prescribed function at mesh points need not exist. As a result, the splines of odd degree do yield an advantageous extension of the cubic splines, but for splines of even degrees, the extension of cubic splines must be modified.

However, as the order of the spline function increases, the algebra involved increases and as a result computation (in numerical applications) become more and more tedious, especially when the mesh integral Δ is comprised of unequal intervals. For equal intervals, the extension up to the order of five is not very tedious. The defining equation of polynomial splines up to the order of seven for the equal interval case has been derived in [6].

2.10. Cardinal Splines

At this point, it is desirable to introduce the cardinal splines $\{W_{\Delta i}, V_{\Delta j}\}$, $i = 0,1,\ldots, N$. $J = 0, N$. The cardinal spline has the property that [26] on the mesh Δ any cubic spline $S_\Delta(x)$ has the representation:

$$S_\Delta(x) = \sum_{i=0}^{N} S_\Delta(x_i)\, W_{\Delta,i}(x) + \sum S_\Delta'(x_j)\, V_{\Delta,j}(x) \quad a \le x \le b \qquad 2.10.1$$

The cardinal splines $W_{\Delta,k}(x)$ $(i = 0, 1, 2, \ldots, N)$ and $V_{\Delta,j}(x)$ $(j = 0, N)$ are cubic splines with

$$W_{\Delta,k}(x_j) = \delta_{k,j} \quad (j = 0, 1, \ldots, N),$$

$$W_{\Delta,k}'(x_i) = 0 \quad (i = 0 \text{ and } N),\ k = 0, 1, \ldots, N. \qquad 2.10.2$$

$$V_{\Delta,k}(x_j) = 0 \quad (j = 0, 1, \ldots, N),\ V_{\Delta,k}'(x_i) = \delta_{k,i} \quad (i = 0 \text{ and } N)$$
$$k = 0, \text{ and } N. \qquad 2.10.3$$

where $\delta_{k,j}$ is the Kronecker delta.

Thus the cardinal splines are a set of independent splines forming a basis for all cubic splines.

CHAPTER 3

IDENTIFICATION OF CONSTANT PARAMETER LUMPED SYSTEMS

In this chapter, the identification schemes for linear constant parameter lumped systems of second, third and fourth order model developed by Balatoni [6], with the application of cubic splines are described.

3.1. Constant Parameter Second Order System Identification

Let the system be described by the equation

$$Ly = y''(t) + a_1 y'(t) + a_2 y(t) = r(t), \quad 0 < t < T \qquad 3.1.1$$

where $y(t)$ exists in $C^2[0,T]$. a_1 and a_2 are the parameters to be identified. If $S_\Delta(t)$ be the cubic spline interpolating to $y(t)$ in the interval $[0,T]$ on the mesh Δ: $0 = t_0 < t_1 < \cdots < t_N = T$, then

$$S_\Delta(y;t) = \sum_{j=0}^{N} \omega_{\Delta,j}(t)\, y(t_j) + y'(t_0)\, v_{\Delta,0}(t) + y'(t_N)\, v_{\Delta,N}(t) \qquad 3.1.2$$

where $\omega_{\Delta,j}$ and $v_{\Delta,j}$ are cardinal splines as described in section 2.10.

Defining the error spline

$$E_\Delta(y;t) = y(t) - S_\Delta(y;t) \qquad 3.1.3$$

equation (3.1.1) can be written

$$LS_\Delta(y;t) - r(t) = S_\Delta''(y;t) + a_1 S_\Delta'(y;t) + a_2 S_\Delta(y;t) - r(t) \qquad 3.1.4$$

Defining

$$G_\Delta(y;t) = - E_\Delta''(y;t) - a_1 E_\Delta'(y;t) - a_2 E_\Delta(y;t) \qquad 3.1.5$$

and substituting in equation (3.1.2) we have

$$LS_\Delta(y;t) - r(t) = G_\Delta(y;t) \qquad 3.1.6$$

Writing $y(t_j) = y_j$ and combining equation (3.1.2) and (3.1.6), we finally get

$$\sum_{j=0}^{N} y_j \, L\omega_{\Delta,j}(t_i) + y_o' \, Lv_{\Delta,0}(t_i) + y_N' \, Lv_{\Delta,N}(t_i) = r(t_i)$$
$$+ G_\Delta(y;t_i) \quad (i = 0, 1, \ldots, N) \qquad 3.1.7$$

Given y_o' and y_N', this equation yields $N + 3$ linear equations in $y_o', y_o, y_1, \ldots, y_{N-1}, y_N, y_N'$. The resultant matrix equation becomes

$$H_\Delta Y_\Delta = R_\Delta + G_\Delta \qquad 3.1.8$$

where $y_\Delta = (y_o', y_o\ y_1, \ldots, y_{N-1}, y_N, y_N')^T$, R_Δ is the input vector at $t_j (J = 0, 1, \ldots, N)$, G_Δ is the error term involving operator L and H_Δ is the coefficient matrix, containing unknown parameters a_1 and a_2.

Since H_Δ^{-1} exists [1], $||H_\Delta^{-1} G_\Delta|| \rightarrow 0$ as $||\Delta|| \longrightarrow 0$ and in the limit equation (3.1.8) becomes

$$H_\Delta Y_\Delta = R_\Delta \qquad 3.1.9$$

So if y_Δ and R_Δ are known, H_Δ can be identified. H_Δ consist of terms such as $L\omega_{\Delta,i}(t_j)$ and $Lv_{\Delta,i}(t_j)$. Premultiplying both sides of equation (3.4.9) by coefficient matrix A (from equation 2.4.1) as shown by Ahlberg [1], H_Δ is made up of expressions arranged in a tridiagonal form with irregularity at first and last row due to boundary conditions. For a general second order system with equal intervals, the i^{th} row of equation (3.1.9) is given by

$$
\begin{bmatrix}
\cdots & & & & \bigcirc \\
 & C_{i,i-1} & C_{i,i} & C_{i,i+1} & \\
\bigcirc & & & & \cdots
\end{bmatrix}
\begin{bmatrix}
\vdots \\ y_{i-1} \\ y_i \\ y_{i+1} \\ \vdots
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\vdots \\
\dfrac{\frac{1}{2} r_{i-1} + 2 r_i + \frac{1}{2} r_{i+1}}{3} \\
\vdots
\end{bmatrix} \qquad 3.1.10
$$

where

$$C_{i,i-1} = \frac{3}{h^2} - \frac{3a_1}{h} + \frac{1}{2} a_2 \qquad 3.1.11(a)$$

$$C_{i,i} = -\frac{6}{h^2} + 2a_2 \qquad 3.1.11(b)$$

$$C_{i,i+1} = \frac{3}{h^2} + \frac{3a_1}{h} + \frac{1}{2} a_2 \qquad 3.1.11(c)$$

So that the recursive equation in the two unknowns can be expressed as

$$
\left[\frac{3}{h^2} - \frac{3a_1}{2h} + \frac{a_2}{2}\right] y_{i-1} + \left[-\frac{6}{h^2} + 2a_2\right] y_i
$$

$$
+ \left[\frac{3}{h^2} + \frac{3a_1}{h} + \frac{a_2}{2}\right] y_{i+1} = \frac{1}{3}\left[\frac{r_{i-1}}{2} + 2r_i + \frac{r_{i-1}}{2}\right] \qquad 3.1.12
$$

Rearranging we get

$$
\left[\frac{y_{i+1} - y_{i-1}}{2h}\right] a_1 + \left[\frac{y_{i+1} + 4 y_i + y_{i-1}}{6}\right] a_2
$$

$$
= \left[\frac{r_{i+1} + 4 r_{i-1} \; r_{i-1}}{6}\right] - \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] \qquad 3.1.13
$$

It is possible to obtain the same result by a straight forward technique. Let $y(t)$ and $r(t)$ be approximated by the following cubic spline construction

$$y(t) \simeq \sum_{i=0}^{N} y(t_i)\ \omega_i(t) \qquad 3.1.14(a)$$

$$r(t) \simeq \sum_{i=0}^{N} r(t_i)\ \omega_i(t) \qquad 3.1.14(b)$$

where $\omega_i(t)$ is cardinal spline as defined in section 2.10. Hence,

$$Ly(t_{j-1}) + 4Ly(t_j) + Ly(t_{j+1}) = \sum_{i=0}^{N} y(t_i) \Big[\{\omega_i''(t_{j-1}) + 4\omega_i''(t_j) + \omega_i''(t_{j+1})\} + a_1 \{\omega_i'(t_{j-1}) + 4\omega_i'(t_j) + \omega_i'(t_{j+1})\} + a_1\{\omega_i(t_{j-1}) + 4\omega_i(t_j) + \omega_i(t_{j+1})\}\Big]$$

$$+ \sum_{i=0}^{N} r(t_i) \left[\omega_i(t_{j-1}) + 4\omega_i(t_j) + \omega_i(t_{j+1})\right] \qquad 3.1.15$$

The equation along with the continuity condition of section 2.4 results

$$\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] + a_1 \left[\frac{y_{i+1} - y_{i-1}}{2h}\right] + a_2 \left[\frac{y_{i+1} + 4y_i + y_{i-1}}{6}\right]$$

$$= \left[\frac{r_{i+1} + 4r_i + r_{i-1}}{6}\right] \qquad 3.1.16$$

which is the same as equation (3.1.13). This equation is the general identification equation to be used for second order systems.

3.2. Recursive Solution of Unknown Parameters

When N measurement values both for input and output are available, equation (3.1.13) yields N-2 algebraic equations containing the unknowns a_1 and a_2. As a result, no unique solution exists. However, there exists a least squares fit solution.

It is also apparent that, since infinite accuracy data is unattainable, a single solution of a_1 and a_2 from a pair of equations will exhibit error. The error will also differ for different pairs of equations. Hence, a least squares fit solution for the unknown parameter is essential.

The recursive least squares estimation procedure as given by Lee [18] can be used effectively, permitting the identification to be on-line. The recursive algorithm is developed using ordinary calculus. The algorithm is stated as follows:

Consider the following m algebraic equations

$$C_i X = z_i \quad (i = 1, \ldots, m) \quad , m > n \qquad 3.2.1$$

where C_i is a n row vector and X is an unknown n column vector. If α' be the row vector formed by the coefficient of X in the (k+1)th equation (i.e. C_{k+1}), the recursive least squares estimate of (k+1)th iteration is given by

$$\hat{X}_{k+1} = X_k + P_k\alpha(\alpha' P_k\alpha + 1)^{-1} (z_{k+1} - \alpha' \hat{X}_k), \; k > n \qquad 3.2.2$$

where $\hat{X}_k$ is the estimate of X after the k^{th} iteration and P_k is given by the following recursive equation

$$P_{k+1} = P_k - P_k\alpha \, (\alpha' P_k \, \alpha+1)^{-1} \alpha' P_k \qquad 3.2.3$$

with $P_n^{-1} = A_n' A_n$, A_n being the coefficient matrix of X with first n equations.

To begin the estimation procedure, the first n equations are solved to get $\hat{X}_n$ and is the initial covariance matrix is $P_n = [A_n' A_n]^{-1}$; then recursive estimation by equation (3.2.2) and (3.2.3) can be carried out.

However, in the limit $k \to \infty$, P_k asymptotically approaches a constant value and with arbitrary choice of $\hat{X}_o$ and $\hat{P}_o$, X was observed to converge rapidly.

Most experiments are conducted in a noisy environment so that the actual observation is given by

$$z_i = y_i + v_i \qquad 3.2.4$$

where y_i is the actual output and v_i is a measurement or system noise. v_i is assumed to be a white gaussian random vector with zero mean. If it is assumed that equation (3.1.13) also holds for z_i, then the equation becomes

$$a_1 \left[\frac{z_{i+1} - z_{i-1}}{2h}\right] + a_2 \left[\frac{z_{i+1} + 4z_i + z_{i-1}}{6}\right] = \left[\frac{r_{i+1} + 4r_i + r_{i-1}}{6}\right] - \left[\frac{z_{i+;} - 2z_i + z_{i-1}}{h^2}\right] \qquad 3.2.5$$

If this equation is used to estimate a_1 and a_2, we must proceed for a stochastic solution in obtaining a consistent estimate. A stochastic interpretation of the recursive algorithm, discussed before, is given by Lee [18], when the measurement vector z_i is contaminated with white random noise. It has been shown by assuming the independence of coefficient matrix A_k and noise vector V_k, that even in the presence of noise the estimate $\hat{X}_k$ is unbiased and equation (3.2.2) and (3.2.3) can be rewritten as (defining σ^2 as the variance of v_k)

$$\hat{X}_{k+1} = \hat{X}_k + \hat{P}_k \quad (\alpha' \hat{P}_k\alpha + \sigma^2)^{-1} \; (z_{k+1} - \alpha' \hat{X}_k) \qquad 3.2.6$$

$$\hat{P}_{k+1} = \hat{P}_k - \hat{P}_k\alpha \; (\alpha' \hat{P}_k\alpha + \sigma^2)^{-1} \; \alpha' \hat{P}_k \qquad 3.2.7$$

which is one form of Kalman estimator in the static case. The only difference is that P_{k+1} is an estimate of the covariance matrix, rather than covariance matrix itself. It is important to note that the covariance matrix P_k always decreases and in the limit approaches zero as $k \to \infty$, so that the estimate is statistically consistent and the same algorithm can be used when the measurement is noise contaminated. Rapid and efficient updating of estimates is possible with this algorithm using a digital computer.

3.3. Recursive Identification of 3rd and 4th Order Model

Consider the third order model

$$Ly = r(t) \qquad 3.3.1$$

where L is given by

$$L = \frac{d^3}{dt^3} + a_1 \frac{d^2}{dt^2} + a_2 \frac{d}{dt} + a_3 \text{ , and} \qquad 3.3.2$$

$y(t)$ exists in C^3 [0, T]. Applying a quartic spline to this model for equal mesh interval, Balatoni [6] obtained the identifying equation as

$$\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right]a_1 + \left[\frac{y_{i+1} - y_{i-1}}{2h}\right]a_2 + \left[\frac{y_{i+1} + 11y_i + 11y_{i-1} + y_{i-2}}{24}\right]a_3 = \left[\frac{r_{i+1} + 11r_i + 11r_{i-1} - r_{i-2}}{24}\right] - \left[\frac{y_{i+1} - 3y_i + 3y_{i-1} - y_{i-2}}{h^3}\right]. \qquad 3.3.3$$

Using this equation with the recursive least squares technique discussed before, the unknown parameters may be extracted. However, appreciable error on the last term of the right hand side is expected, particularly when the measurement data is noise corrupted. As an alternate procedure of identification, if the process input and output are integrable, then application of cubic spline results in the following identifying equation

$$\left[\frac{y_{i+1} - y_{i-1}}{2h}\right]a_1 + \left[\frac{y_{i+1} + 4y_i + y_{i-1}}{6}\right]a_2 + \left[\frac{\bar{y}_{i+1} + 4\bar{y}_i + \bar{y}_{i-1}}{6}\right]a_3$$

$$= \left[\frac{\bar{r}_{i+1} + 4\bar{r}_i + \bar{r}_{i-1}}{6}\right] - \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] \qquad 3.3.5$$

where $\bar{y} = \int_0^{t_i} y(t)\,dt$ and $\bar{r}_i = \int_0^{t_i} r(t)\,dt$ 3.3.6

The use of this equation with the recursive least squares technique to estimate the process parameters, yields more accurate and rapidly converging results then is possible by the use of quartic splines.

Consider the fourth order model

$$Ly = r(t), 0 < x < T \qquad 3.3.7$$

where L is given by

$$L = \frac{d^4}{dt^4} + a_1\frac{d^3}{dt^3} + a_2\frac{d^2}{dt^2} + a_3\frac{d}{dt} + a_4 \qquad 3.3.8$$

$y(t)$ exists in C^4 [0, T], then for equal length of interval and applying quintic spline, the resulting equation becomes

$$\left[\frac{y_{i+1} - 3y_i + 3y_{i-1} - y_{i-2}}{h^3}\right]a_1 + \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right]a_2$$

$$+ \left[\frac{y_{i+1} - y_{i-1}}{2h}\right]a_3 + \left[\frac{y_{i+2} + 26y_{i+1} + 66y_i - 26y_{i-1} + y_{i-2}}{120}\right]a_4$$

$$= \left[\frac{r_{i+2} + 26v_{i+1} + 66r_i + 26r_{i-1} + r_{i-2}}{120}\right]$$

$$- \left[\frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4}\right] \qquad 3.3.9$$

If higher order splines are used,the identifying equation becomes more complex. Since h_i, is small, the last term of the right hand side contributes appreciable errors especially when the measurements are noisy.

Using an alternate procedure, assuming that $\int y(t)\,dt$, $\iint y(t)\,dt^2$ and $\iint r(t)\,dt^2$ are available as observation data or can be evaluated, the identification can be achieved by the use of cubic splines. Application of cubic splines results in the identifying equation

$$\left|\frac{y_{i+1} - y_{i-1}}{2h}\right| a_1 + \left|\frac{y_{i+1} + 4y_i + y_{i-1}}{6}\right| a_2 + \left|\frac{\bar{y}_{i+1} + 4\bar{y}_i + \bar{y}_{i-1}}{6}\right| a_3$$

$$+ \left|\frac{\bar{\bar{y}}_{i+1} + 4\bar{\bar{y}}_i + \bar{\bar{y}}_{i-1}}{6}\right| a_4 = \left|\frac{\bar{\bar{r}}_{i+1} + 4\bar{\bar{r}}_i + \bar{\bar{r}}_{i-1}}{6}\right|$$

$$- \left|\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right| \qquad 3.3.10$$

where

$$\bar{y}_i = \int_0^{t_i} y(t)\,dt \qquad 3.3.11(a)$$

$$\bar{\bar{y}}_i = \int_0^{t_i}\int_0^{t_i} y(t)\,dt^2 \qquad 3.3.11(b)$$

$$\bar{\bar{r}}_i = \int_0^{t_i}\int_0^{t_i} r(t)\,dt^2 \qquad 3.3.11(c)$$

As before, this equation when used for identifying a fourth order model, results in more accurate parameter estimation and faster convergence, compared to the application of equation (3.3.9).

3.4. Model Adjustment Method

Equation (3.1.13) can be rearranged to get it in the form

$$\hat{y}_{i+1} = f(y_i, y_{i-1}, r_i, r_{i-1}, a_1, a_2) \qquad 3.4.1$$

so that if $y_o = z_o$, $y_1 = z_1$ are given with initial estimates $\hat{a}_1$ and $\hat{a}_2$ respectively, then y_i can be generated for $i = 2, 3, \ldots, N$. The performance index

$$J = \frac{1}{N} \sum (z_i - \hat{y}_i)^2 \qquad 3.4.2$$

is computed where z_i is the actual observation. An optimization scheme can be used to adjust $\hat{a}_1$ and $\hat{a}_2$ to minimize J. However, the technique has the disadvantage that it is off-line.

CHAPTER 4

LINEAR TIME VARYING AND NON-LINEAR SYSTEM IDENTIFICATION

An identification scheme for systems described by linear time varying differential equation is described here. The identification scheme for a class of non-linear systems is also presented. Lastly, a general identification procedure using the state variable approach, applicable to multiple input/output systems, is described. The effect of noisy measurements on the accuracy of parameter estimates is also considered.

4.1. Second Order Time Varying Systems

Consider a linear lumped parameter system described by

$$Ly(t) = v(t)$$

where $L = \frac{d^2}{dt^2} + a_1(t)\frac{d}{dt} + a_2(t)$

$y(t)$ exists in $C^2(0,T)$. $a_1(t)$ and $a_2(t)$ are the time varying parameters to be estimated.

If the sampling interval h is small enough so that for three successive samples $a_1(t)$ and $a_2(t)$ can be assumed to be approximately constant, then equation (3.1.13) derived for a constant parameter system also holds for a time varying system. Then

$$a_{1i}\left[\frac{y_{i+1} - y_{i-1}}{2h}\right] + a_{2i}\left[\frac{y_{i+1} + 4y_i + y_{i-1}}{6}\right]$$
$$\approx \left[\frac{r_{i+1} + 4r_i + r_{i-1}}{6}\right] - \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] \qquad 4.1.1.$$

where $a_{ji} = a_j(t_i)$, $(j = 1, 2. \quad i = 1, 2, \ldots, N)$

This equation is the general identifying equation to be used for a time varying linear second order system on a uniform mesh. The solution technique for estimating the unknown dynamic parameters will now be discussed.

It is important to note that it is not necessary to consider the boundary conditions, so that the identification can be carried out on-line and for any unknown end points.

4.2. Recursive Solution of Time Varying Parameters

Since neither infinite precision data nor infinite precision computation is attainable in practice, the solution of each pair of equations will exhibit some error. As a result, a least squared error of the parameters is essential.

A recursive solution algorithm, suitable for implementation on a digital computer, for estimating a vector X from a set of linear algebraic equations relating X is given by Lee [18]. The error solution procedure may be stated as follows.

Let X and z be governed by the dynamic relationship

$$X_{k+1} = \phi X_k \tag{4.2.1}$$

$$z_k = H' X_k \tag{4.2.2}$$

where ϕ and H may be time varying and it is desired to estimate X_{k+1}, when z_i and the corresponding H are given for $i = 1,2,\ldots, k+1$. It can be shown that the least squares estimate of X is given by

$$\hat{X}_{k+1/k+1} = \phi X_{k/k} + M_{k+1} H(H'M_{k+1} H'+I)^{-1} (z_{k+1} - H'\phi \hat{X}_{k/k}) \tag{4.2.3}$$

where $X_{k/k}$ implies the estimate of X at time k given measurements up to and including z_k. M_{k+1} is given by the recursive relation

$$M_{k+1} = \phi P_k \phi' \tag{4.2.4}$$

with $\hat{P}_{k+1} = M_{k+1} - M_{k+1} H(H'M_{k+1}H + I)^{-1} H'M_{k+1}$ 4.2.5.

and $P_n = A_n' A_n$, where n is the dimension of X and A_n is the coefficient matrix formed by equation (4.2.2), k=1, ..., n. $X_{n/n}$ can also be found by simply solving the set of equations (4.2.2) for k = 1, 2, ..., n. However, with an arbitrary choice of $X_{o/o}$ and P_o the estimate of X has been observed to converge within reasonable duration.

Lee also gave a stochastic interpretation of equation (4.2.3), when observation z_k is contaminated with white noise, i.e.,

$$z_k = H'Z_k + v_k \qquad 4.2.6$$

where v_k is a white noise with zero mean and variance σ^2. It has been shown that equations (4.2.3) - (4.2.6) are precisely Kalman estimation formulas. The only difference is that $P_{k+1/k}$ is the estimation of the covariance matrix rather than the covariance matrix itself. It can then be concluded that equations (4.2.3. - 4.2.5) also hold when observations are corrupted by noise, and the solution is consistent.

4.3. Parameter Variation of Unknown Nature

In the previous section, we assumed that $X_{k+1} \triangleq \phi X_k$, where ϕ was known. In this section the case when parameter variation with time is not known apriori will be considered. Specifically, this implies that ϕ is not known.

(i) In one attempt to find ϕ_k, the parameter values at the $(k+1)^{th}$ iteration may be estimated as lying on a polynomial passing through the previously identified values. As polynomial approximation is excessively oscillatory, the procedure would be very much sensitive to noise and might cause the recursive algorithm to be unstable.

As an alternate procedure a least squares straight line fit to the previous n identified parameter values may be used to estimate the new parameter value. However, if the parameter computations at $(k-\ell)^{th}$ iteration ($\ell = 1, 2, \ldots, n$) are very different from actual values and are biased in the same direction (due to initial guess, noise, etc.) the error in the new estimates may increase and the resulting identification will be unacceptable. This difficulty can be eliminated by estimating the parameter values to be on a least square straight line passing through some crude parameter estimates which are not calculated recursively.

To estimate ϕ_k, each pair of $(k-\ell+1)^{th}$ and $(k-\ell)^{th}$ equations (in case of two parameters) can be solved (for $\ell = 1, 2, \ldots, n$) to get a set of n crude parameter estimates corresponding to $(k-\ell+1)^{th}$ iteration. These n crude estimates can be stored in an arbitrary array. A least squares straight line can be passed through these n computed values and the parameter value at $(k+1)^{th}$ iteration may be estimated to lie on this least squares straight line.

An improved estimate may now be obtained from the identifying equation at $(k+1)^{th}$ iteration, using the recursive least squares algorithm. Since the crude estimate stored for getting the least squared fit straight lines were not obtained in a recursive manner, the new estimates tend to follow the actual parameter variations more accurately.

This gives a physical interpretation that parameter variation is piecewise linear. However, it is apparent that the procedure requires elaborate computational effort.

(2) When the parameter variation is relatively slow in comparison to

the convergence rate of the identification procedure, the parameter can be assumed to be piecewise constant and equation (3.2.2) derived for the static case can be applied stepwise. The identification procedure can be stated as follows:

"Start with an initial estimate X_o and P_o compute for $X_\ell(\ell>0)$ along equation (3.2.2). If from previous experience it is known that the algorithm converges to within an acceptable limit in m iterations, then reinitiate the identification procedure after every m iteration with same value of P_o and last estimated $\hat{X}_m$."

The physical interpretation is that the procedure places more weight on current data and prevents P to approach zero by updating the covariance matrix every m iterations, so that the current estimate always tries to track the slow parameter variations. However, this procedure is neither optimal nor consistent; there is no assurance of convergence. Its application also depends upon the particular problem and the assumption of slow parameter variation.

(3) As ϕ is not known, it can be assumed that $\phi = I$ and the algorithm given in section 4.2 can be used with the modification.

$$M_{k+1} = \phi P_k \phi' + C. \qquad 4.3.1$$

where C is an arbitrary matrix chosen by experimentation.

This gives a physical interpretation that the parameter variation is random. In the computation process the matrix M_k in the limit approaches a constant value and the estimating procedure will always tend to follow the actual variation. The weighting term of the residues will not reduce to zero. It is important to note that the scheme is not optimal in any sense, the procedure is purely intuitive having no assurance of convergence. The only justification is that "it works". The usefulness and choice of C is determined purely by experimentation and will be discussed later.

4.4. Higher Order Systems

Consider the third order model

$$Ly(t) = r(t) \qquad 0 < t < T \tag{4.4.1}$$

where

$$L = \frac{d^3}{dt^3} + a_1(t)\frac{d^2}{dt^2} + a_2(t)\frac{d}{dt} + a_3(t) \tag{4.4.2}$$

$y(t)$ exists in $C^3\,[0,T]$, and the time varying parameters $a_1(t)$, $a_2(t)$ and $a_3(t)$ to be identified. For equal mesh interval on $[0,T]$ with small h, if $a_1(t)$, $a_2(t)$ and $a_3(t)$ can be assumed to be approximately constant for four successive mesh points, then as shown in section 3.3, application of quartic splines yields the identifying equation

$$\left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] a_{1i} + \left[\frac{y_{i+1} - y_{i-1}}{2h}\right] a_{2i}$$
$$+ \left[\frac{y_{i+1} + 11y_i + 11y_{i-1} + y_{i-2}}{24}\right] a_{3i} = \left[\frac{r_{i+1} + 11r_i + 11r_{i-1} + r_{i-2}}{24}\right]$$
$$- \left[\frac{y_{i+1} - 3y_i + 3y_{i-1} - y_{i-2}}{h^3}\right] \tag{4.4.3}$$

where $a_{ji} = a_j(t_i)$, $(j = 1, \ldots, 3)$

Consider the 4th order model

$$Ly(t) = r(t) \qquad 0 < t < T \tag{4.4.4}$$

$$L = \frac{d^4}{dt^4} + a_1(t)\frac{d^3}{dt^3} + a_2(t)\frac{d^2}{dt^2} + a_3(t)\frac{d}{dt} + a_4 \tag{4.4.5}$$

$y(t)$ exists in $C^4\,[0,T]$. Considering the case of equal mesh intervals and approximating the parameters to be constant for five successive mesh points, application of quintic spline results in the following equation

$$\left[\frac{y_{i+1} - 3y_i + 3y_{i-1} - y_{i-2}}{h^3}\right]a_{1i} + \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right]$$

$$+\left[\frac{y_{i+1} - y_{i-1}}{2h}\right]a_{3i} + \left[\frac{y_{i+2} + 26y_{i+1} + 66y_i + 26y_{i-1} + y_{i-2}}{120}\right]a_{4i}$$

$$=\left[\frac{r_{i+2} + 26r_{i+1} + 66ry_i + 26y_{i-1} + r_{i-2}}{120}\right]$$

$$-\left[\frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4}\right] \qquad 4.4.6$$

This set of equations together with the recursive least squares algorithm described in section 4.2 and 4.3 can be applied for identification of 3rd and 4th order models.

However, the terms containing higher powers of h in the denominator might cause appreciable error, especially with noisy data. We have also seen that higher mesh intervals are preferred in noisy environments and with a higher value of h the assumption of parameters to be constant for five successive mesh points (for fourth order model) may not hold. Moreover the previous experience [6] has shown that for a constant parameter model, identification by cubic splines is more accurate and rapidly converging compared to the application of quartic and quintic splines. A more efficient identification scheme for higher order systems can be realized by utilizing the 'spline on spline technique of derivative generation' with the help of cubic splines.

4.5. Spline on Spline Identification Technique

As the very name implies, in this technique the output function together with its derivatives are approximated by cubic splines to calculate the immediate higher derivatives. As discussed in section 2.3, the first derivatives of spline functions are related by the expressions

$$\lambda_j M_{j-1} + 2M_j + \mu_j M_{j+1} = 3\lambda_j \frac{y_j - y_{j-1}}{h_j} + 3\mu_j \frac{y_{j+1} - y_j}{h_{j+1}}$$

for $\quad j = 2, 3, \ldots, N-1$ 4.5.1

with λ_j and μ_j as defined in section 2.3.

With equal mesh intervals this reduces to

$$\frac{1}{2} M_{j-1} + 2M_j + \frac{1}{2} M_{j+1} = \frac{3}{2} \frac{(y_{j+1} - y_{j-1})}{h} \qquad 4.5.2$$

or $$M_{j+1} = 3 \frac{(y_{j+1} - y_{j-1})}{h} - 4M_j - M_{j-1} \qquad 4.5.3$$

However, the difference equation is unstable, as one of its poles lies outside the unit circle and recursive realization of M_{j+1} with arbitrary end condition is not feasible.

A very efficient algorithm which is available for solving the system of equations (4.5.1) is given by Ahlberg [1], which may be stated as follows.

Given the equations

$$\begin{aligned} b_1x_1 + c_1x_2 &= d_1 \\ a_2x_1 + b_2x_2 + c_2x_3 &= d_2 \\ a_3x_2 + b_3x_3 + c_3x_4 &= d_3 \qquad 4.5.4 \\ &\cdots\cdots \\ &\cdots\cdots \\ a_{n-1}x_{n-2} + b_{n-1}x_{n-1} + b_nx_n &= d_{n-1} \end{aligned}$$

$$a_n x_{n-1} + b_n x_n = d_n$$

we define (from $k = 1, 2, \ldots, n$)

$$p_k = a_k q_{k-1} + b_k \quad (q_0 = 0) \qquad 4.5.5$$

$$q_k = - C_k / p_k \qquad 4.5.6$$

$$u_k = (d_k - a_k u_{k-1}) / p_k \quad (u_0 = 0) \qquad 4.5.7$$

Successive elimination of $x_1, x_2, \ldots, x_{n-1}$ from 2nd, 3rd, ..., nth equations yields the equivalent system

$$x_k = q_k x_{k+1} + u_k \quad (k = 1, \ldots, n-1) \qquad 4.5.8$$

$$x_n = u_n \qquad 4.5.9$$

Thus $x_n, x_{n-1}, \ldots, x_1$ can be evaluated in a backward recursive way. The procedure is stable in the sense that errors rapidly damp out ($0 < C_k/p_k < 1$). Also quantities p_k and q_k in the spline application depend upon the mesh Δ and not on mesh ordinates. These can then be precalculated and can be used for several spline applications on the same mesh. However, this scheme is no longer on-line, since the derivative estimation is achieved by solving the system of equations in a backward recursive manner.

As discussed before, the choice of end conditions affects the derivative estimates at the end points. However, this effect is not significant as one moves from the extremeties, so any specific end conditions as discussed as in section 2.4 can be applied. In this study, the choice was, $M_0 = M_1$ and $M_{N-1} = M_N$. For equal mesh intervals the backward recursive solution technique yields interesting results. With end conditions as discussed, from equation (4.5.2), one can write

$$
\begin{aligned}
&a_1 = 0 \\
&a_i = 1/2 && (i = 2, 3, \ldots, N) \\
&b_1 = b_n = 5/2 \\
&b_i = 2 && (i = 2, 3, \ldots, N-1) \qquad 4.5.10 \\
&C_i = 1/2 && (i = 1, 2, \ldots, N-1) \\
&C_N = 0
\end{aligned}
$$

Substituting these values in equation (4.5.5) results

$$
\begin{aligned}
p_1 &= 2.500 \\
p_2 &= 1.900 \\
p_3 &= 1.868 \\
p_4 &= 1.866 \\
p_5 &= 1.866 \qquad 4.5.11 \\
&\cdots \\
p_{N-2} &= 1.866 \\
p_{N-1} &= 1.866 \\
p_N &= 2.366
\end{aligned}
$$

Thus p_k asympotically reaches a constant value of 1.866 (except p_N) and as a result, q_k also reaches a constant value of 0.268. When these values of p_k along with the equation (4.5.8) were used to evaluate the quantities m_j ($j = 0, 1, \ldots, N$), the derivatives calculated at mesh points x_i ($i = 1, 2, 3$ and $N-2, N-1, N$) near extremities were apart from the actual derivative of the function interpolated, but the derivatives at mesh points x_j ($J = N-3, N-2, \ldots, 4$) were within acceptable accuracy.

This suggests another on-line recursive identification procedure. Since P_k and q_k reaches asymptotically constant values and computed values of m_{N-3} are acceptable, one can take every k^{th} sample ($k=7,8,\ldots, N$), calculate u_k by equation (4.5.7), compute m_i ($i = k, k-1, k-2$) by equation (4.5.8) and use the value m_{k-3} in the identifying equation. If the system is second order, the second derivative can be immediately calculated with the help of equation (2.4.8). However, if the system is of a higher order, then by fitting another spline through the derivatives, the same procedure is followed.

This procedure removes the restriction of boundary conditions; hence, the identification can be carried out on-line and for any undefined end points.

The derivatives thus calculated when applied to the system equation, is given by

$$\sum_{i=1}^{\ell} c_i a_i = r_i \qquad (r_i \text{ is a known system input}) \qquad 4.5.12$$

Where ℓ equals the order of the systems and c_i is known. a_i ($i=1,2,\ldots, \ell$) can be solved by a recursive least squares algorithm. It is important to note that this procedure can be applied to systems both stationary and nonstationary. It is also applicable to a class of non-linear systems.

4.6. Identification of Nonlinear Systems

As discussed in the previous section, the recursive identification, after computing the derivative by application of cubic splines, is also applicable to a restricted class of nonlinear systems. If the order of the nonlinear system is two or less, then the method described in the previous section can be applied to estimate the first and second derivatives directly. Also if the differential equation describing the nonlinear system is 'linear in parameter', the resulting algebraic equation relating the unknown parameters are linear and may be solved by the recursive least square technique. For higher order nonlinear systems, the 'spline on spline' technique may be used.

For one particular class of nonlinear systems described by the form of equation

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_2 y^s = r(t) \qquad 4.6.1$$

where y exists in C^2 [0,T], the application of cubic splines results in the closed form identifying equation:

$$a_1\left[\frac{y_{i+1} - y_{i-1}}{2h}\right] + a_2\left[\frac{y_{i+1}^s + 4y_i^s + y_{i-1}^s}{6}\right]$$

$$= \left[\frac{r_{i+1} + 4r_i + r_{i-1}}{6}\right] - \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] \qquad 4.6.2$$

This is quite similar to the closed form equation obtained for 2nd order linear system and can be solved by a least squares technique for the unknown process parameters a_1 and a_2.

4.7. State Variable Approach

Linear systems:

Consider a multivariable linear process defined by

$$\dot{X} = AX + BU \qquad 4.7.1$$

$$Y = CX + DV \qquad 4.7.2$$

where

A is a n x n matrix

B is a n x r matrix

C is a k x n matrix

D is a k x l matrix

X is a n vector system state variable

U is the r dimensional input vector

Y is the measurement vector

V is a noise vector assumed to be zero mean and uncorrelated.

If U is assumed to be a solution of a set of homogeneous equations, then by defining r additional state variables, equation (4.7.1) and (4.7.2) can be expressed as

$$\dot{X}' = \phi X' \qquad 4.7.3$$

$$Y' = CX' + DV \qquad 4.7.4$$

The application of cubic spline approximation to X' results in

$$\frac{1}{2h}[X'(i+1) - X'(i-1)] = \frac{1}{6}\phi[X'(i+1) + 4x'(i) + X'(i-1)] \qquad 4.7.5$$

Depending on the nature of the problem, the solution procedure for matrix A may take different forms.

(a) If the measurement Y is an (nx1) vector and n outputs of the process are available with matrix C known, then the state variables can be found by solving the system of equations, i.e.

$$X = C^{-1}Y \qquad 4.7.6$$

and the equation

$$\frac{1}{2h}[X(i+1) - X(i-1)] = \frac{1}{6}\phi[X'(i+1) + 4X'(i) + X'(i-1)] \qquad 4.7.7$$

results in a system of algebraic equations which may be solved by recursive least squares procedure to get n x n element of A

(B and U are assumed to be known).

(b) When n distinct output are not available as observations, state variables can be estimated only if matrix A is partially known. The solution procedure in the special case when one output is available and A is in canonic form with at most n unknown quantities to be identified in the matrix, will now be discussed.

The canonic form of a third order system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t), \quad y = x_1 \qquad 4.7.8$$

where y is a scalar. The identification procedure involves the estimation of state variables using splines. As suggested by Balatoni [6], with $y = x_1$ begin at $j = 1$ and solve for $x_{j+1}(i)$ using equation (4.7.8) and

$$\dot{x}_i(i) = x_{j+1}(i) \qquad 4.7.9$$

By application of cubic splines on a uniform mesh this becomes

$$\frac{1}{2h}[x_j(i+1) - x_j(i-1)] = \frac{1}{6}[x_{j+1}(i+1) + 4x_{j+1}(i) + x_{j+1}(i-1)] \qquad 4.7.10$$

which forms a system of N-1 equation in the N+1 unknowns $X_{j+1}(k)$, $k = 0, 1, \ldots, N$. To estimate all N+1 unknowns, we need equations concerning end conditions. The following may be used:

$$\text{at } k = 0, \ \frac{1}{h}[x_j(1) - X_j(0)] = \frac{1}{6}[2X_{j+1}(1) + 4X_{j+1}(0)] \qquad 4.7.11(a)$$

$$\text{at } k = N, \ \frac{1}{h}[X_j(N) - X_j(N-1)] = \frac{1}{6}[4X_{j+1}(N) + 2X_{j+1}(N-1)] \qquad 4.7.11(b)$$

The preceding three equations can be expressed in matrix form

$$DX_j = EX_{j+1} \qquad 4.7.12$$

and the solution

$$X_{j+1} = E^{-1} DX_j \qquad 4.7.13$$

is straightforward. Matrix D and E are constant and known. E is diagonally dominant, positive definite and due to it's tridiagonal nature, E^{-1} exists. A simple algorithm to calculate elements of E^{-1} for equal interval is discussed in section 2.6. Thus X_{j+1} is readily generated over the interval of interest (for j = 1, 2, ..., m-1), where m is the order of the system. It is important to note that for a fixed mesh E^{-1} D can be precomputed and needs to be computed only once.

As x_2 and x_3 can be calculated as above, the third order process parameters may be identified using the following equation

$$\frac{1}{2h}[\, x_3(i+1) - x_3(i-1)\,] = -\frac{a_3}{6}[\, x_1(i+1) + 4x_1(i) + x_1(i-1)\,]$$

$$-\frac{a_2}{6}[\, x_2(i+1) + 4x_2(i) + x_2(i-1)\,]$$

$$-\frac{a_1}{6}[\, x_3(i+1) + 4x_3(i) + x_3(i-1)\,] + \frac{1}{6}[r(i+1) + 4r(i) + r(i-1)\,] \qquad 4.7.14$$

with recursive least squares solution algorithm.

(3) When matrix A is not in canonic form, but contains at most n unknown elements, it may still be identified. In such cases, the canonical form is first identified as before and with a_1, a_2, ..., a_n the characteristic equation

$$\lambda^n + a_1 \lambda^{N-1} + \cdots a_n = 0 \qquad 4.7.15$$

is solved to get $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$.

The solution of

$$|A - \lambda I| = 0 \qquad 4.7.16$$

yields the unknown elements in A by using

$\lambda = \lambda_1, \lambda_2, \cdots, \lambda_n$.

Nonlinear Systems:

Similarly, nonlinear systems described by a set of first order differential equations can be written in the form

$$\dot{X} = G + BU \tag{4.7.17}$$

$$Y = CX + DV \tag{4.7.18}$$

where, G is a n column matrix containing the nonlinear terms and the other matrices have the same explanation as given with equations (4.7.1 and 4.7.2).

The spline identification procedure discussed for systems described by linear state equations can be extended for some particular type of nonlinear estimation problem.

(a) When all state variables are available as observations and if matrix C is known, the state variables can be found with the help of equation (4.7.6). Substitution of these values in equation (4.7.8) and approximating all the state variables by cubic splines results in a system of linear algebraic equations containing the unknown parameters which may be solved by the recursive least squares procedure.

(b) When n distinct outputs are not available as observations, state variables can be estimated only if the elements G_i $(i = 1, 2, \ldots, n-1)$ are state variables themselves, and nonlinearity exists in the last element of G(i.e., G_n).

As an example, consider the third order nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ -\gamma x_1 - \beta x_2 x_3 - \alpha x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \gamma(t), \ y = x_1 \tag{4.7.19}$$

where α, β and γ are to be identified and only one measurement x_1 is available.

When matrix G is of this particular form, equation (4.7.13) can be successfully applied in evaluating the other state variables x_2 and x_3. Once x_2 and x_3 are computed, the application of cubic splines gives the identifying equation

$$\frac{1}{2h}[x_3(i+1) - x_3(i-1)] = -\frac{\gamma}{6}[x_1(i+1) + 4x_1(i) + x_1(i-1)$$

$$-\frac{\beta}{6}[x_2(i+1)\,x_3(i+1) + 4x_2(i)\,x_3(i) + x_2(i-1)\,x_3(i-1)]$$

$$-\frac{\alpha}{6}[x_3(i+1) + 4x_3(i) + x_3(i-1)] + \frac{1}{6}[\gamma(i+1) + 4\gamma(i) + \gamma(i-1)]$$

which may be solved by the least squares algorithm, to get the unknown parameters α, β and γ.

The procedure of equations is similar to 'spline on spline' application of derivative generation. Systems of higher order taken in this form can be effectively identified. The procedure generates smooth values of state variables and is therefore especially suitable when the data is only approximate.

4.8. Important Features of Identification by Cubic Splines

The basic equation for linear system identification is equation (3.1.13) which relates the unknown parameter with the input and output of the system. The observed output of the system to be identified may be written as

$$z_i = y_i + v_i \qquad 4.8.1$$

where y_i is the actual output, z_i is the observed output and v_i is measurement noise. We assume that v_i is an uncorrelated gaussian random variable with variance σ^2, so the identifying equation (3.1.13) with y_i replaced by z_i becomes

$$a_1 \left[\frac{y_{i+1} - y_{i-1}}{2h}\right] + a_2 \left[\frac{y_{i+1} + 4y_i + y_{i-1}}{6}\right]$$

$$= \left[\frac{y_{i+1} + 4y_i + y_{i-1}}{6}\right] - \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] - u_i \qquad 4.8.2$$

$$\text{where } u_i = \left[\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}\right] + a_1 \left[\frac{v_{i+1} - v_{i-1}}{2h}\right]$$

$$+ a_2 \left[\frac{v_{i+1} + 4v_i + v_{i-1}}{6}\right] \qquad 4.8.3$$

and the variance of u_i is given by

$$\left(\frac{6}{h^4} + \frac{a_1}{2h^2} + \frac{a_2}{2}\right)\sigma^2 . \qquad 4.8.3(a)$$

As h is small, the error associated with the coefficients of a_1 and a_2 are small compared to the error associated with the first term of equation (4.8.3).

If instead of splines, central difference approximations were used for derivative estimation in identifying parameters of second order systems, the following equation would result

$$a_1 \left[\frac{y_{i+1} - y_{i-1}}{2h}\right] + a_2 y_i = Y_i - \left[\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\right] - u'_i$$

4.8.4

where u'_i consists of noise terms v_i with a variance of

$$\left(\frac{6}{h^4} + \frac{a_1}{2h^2} + a_2\right)\sigma^2 \qquad 4.8.4(a)$$

A comparison of equation (4.8.3) and (4.8.4) reveals the essential difference between the application of splines and the central difference approximations in identification. If the noisy measurement z_i has a variance of σ^2, the noise level associated with the identifying equation by spline approximation and central difference approximation is given by (4.8.3(a)) and (4.8.4(a)) respectively. The smaller noise variance of u_i compared to u_i' causes spline approximation to yield more accurate results than the central difference scheme.

Spline approximation is more accurate in the absence of measurement noise due to the continuity conditions imposed on the first and second derivatives at the mesh points, whereas no such condition exists for the central difference method.

The continuity condition of derivatives also results in more accurate identification than the central difference method at higher values of difference interval h.

An important fact to note here is, that though measurement noise v_i is uncorrelated, the covariance of the noise term u_i is not zero and as a result, u_i is not uncorrelated.

$$E\left(u_i, u_{i+1}\right) = \left(\frac{4}{h^4} + 8\right) \sigma^2 \qquad (i = 0, 1, \ldots, N) \qquad 4.8.5(a)$$

$$E\left(u_i, u_{i+2}\right) = \left(\frac{1}{h^4} + \frac{1}{4h^2} + 1\right)\sigma^2 \qquad (i = 0, 1, \ldots, N) \qquad 4.8.5(b)$$

So the application of the recursive least square algorithm of Lee[18] which assumes the noise u_i to be uncorrelated, must show a bias, especially at moderate to high noise level. However, this bias can be reduced considerably at moderate noise level by taking every third identifying equation. (i.e. using equation (3.1.13) for $i = 1, 4, 7, 10, \ldots$), as we see that

$$E\left(u_i, u_{i+3}\right) = 0 \qquad 4.8.6$$

In the cases where identification schemes involve derivative generation and substitution in the system equation, the advantage of cubic spline over other finite difference methods may be explained as the smoothing of derivatives on adjacent meshes.

This smoothing of derivatives is very much desirable in generating derivatives when the data is known only approximately.

CHAPTER 5

COMPUTER SIMULATED RESULTS

5.1. Identification of Linear Time Varying Systems

The identification schemes generated in the previous chapter were simulated on an IBM 360/65 digital computer. A number of numerical examples were used in the investigation. Initially, the system output was obtained by solving the system equations numerically with the help of Adams-Moulton Predictor-Corrector formula [24], on the digital computer.

In the first example, the system to be identified is given by

$$\ddot{y} + a_1(t)\dot{y} + a_2(t)y = r(t) \tag{5.1.1}$$

where theoretically $a_1(t) = 1 + \cos(0.4t)$, $a_2(t) = 3$ and $r(t) = \sin(t)$. The knowledge that parameter a_1 is time varying and a_2 is stationary was assumed to be known, but the nature of variation was assumed to be unknown. The identification procedure was tested for the various assumptions about parameter variation, as discussed in chapter 4.

(a) Parameter assumed to be piecewise linear:

When the schemes discussed in section 4.3.1, along with equation (4.1.1) was applied, the estimated parameters converged rapidly as illustrated in Figure 1 indicating the feasibility of the procedure. The identification problem was also investigated for additive noise on output observation data. The noise added was wideband gaussian with zero mean. Figure 2 shows the effect of noise level on parameter estimation, plotted as mean squared error (M.S.E.) of the time varying parameter (i.e., a_1) for different noise variances. Mean squared error was defined as

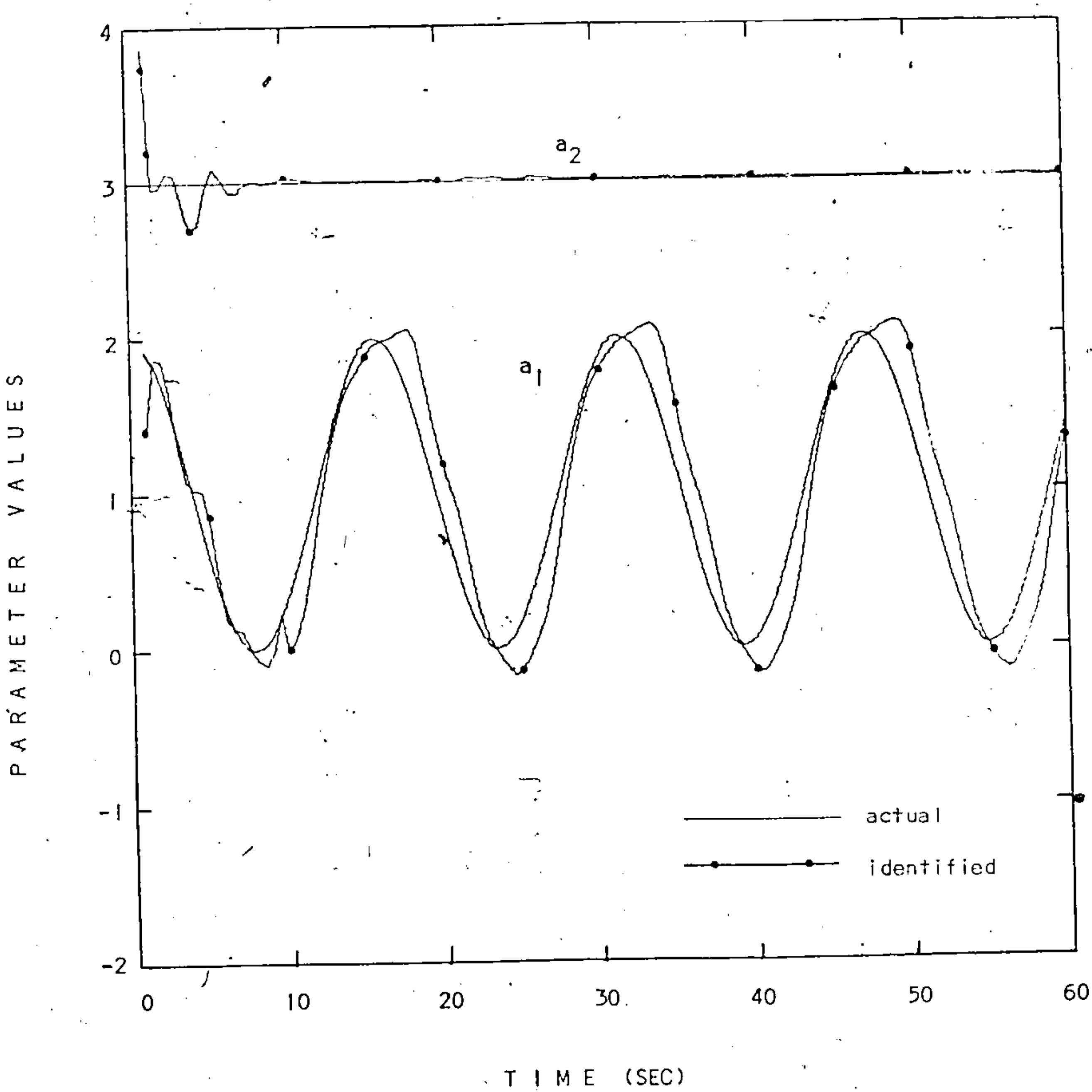

FIGURE 1: Parameter Tracking - Estimating Parameters to be Piecewise Linear

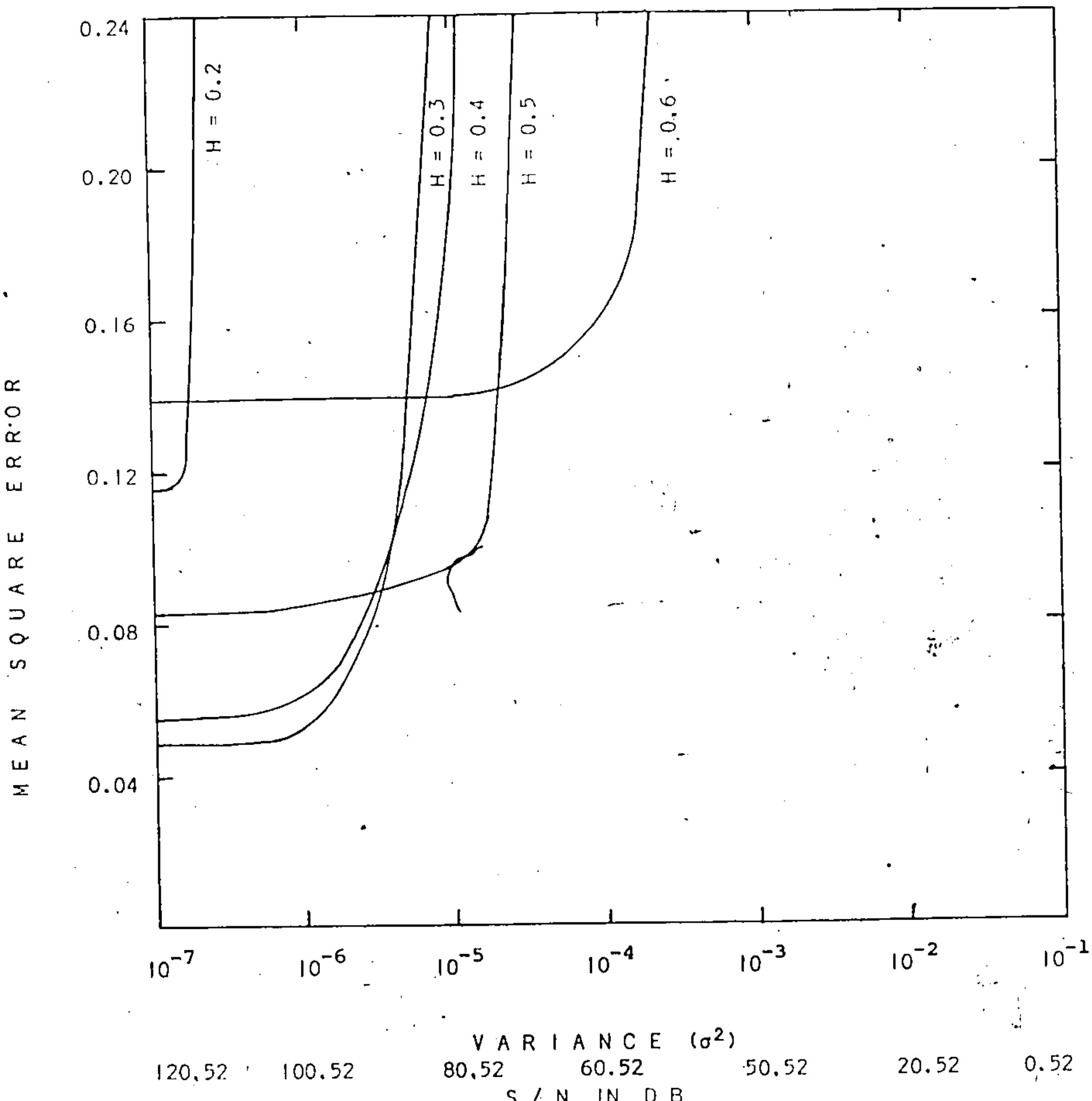

FIGURE 2: Effect of Noise Levels on Nonstationary Parameter Estimation

Assumed Piecewise Linear

In this thesis the Signal to Noise ratio is defined as 20 $\log_{10} \frac{\overline{S^2}}{\sigma^2}$.

$$\frac{1}{N} \sum_{i=1}^{N} (a_i - a_{ie})^2 \qquad 5.1.2$$

where a_i — the estimated value of the parameter

a_{ie} — the theoretical value of the parameter

N — the total number of mesh points.

The error associated with the stationary parameter (i.e. a_2) was small comparatively, and therefore, it is not shown.

(b) Parameter assumed to be piecewise constant:

The same example was also identified using the procedure discussed in section 4.3.2, ie, by changing the covariance matrix P every m iterations. The resulting identification is illustrated in Figure 3. The value of m was assumed to be 10 and P_o used was 100I, where I is the identity matrix. The behaviour of the identification algorithm in the presence of noise with output data was also investigated. Figure 4 shows the mean squared error in estimation of parameter a_1 for different variances of additive noise in the output, using different values of P_o.

The optimum value of P_o depended upon a number of factors, such as rate of variation of the parameter, level of noise in output data, difference interval h, etc. When the parameter variation is rapid, a higher value of P_o was found to be desirable so as to give sufficient weight to the current data. When the noise variance associated with the output data is large, a smaller value of P_o yielded better results. This is illustrated in Figure 4. Similarly, a larger value of P_o is preferred if the difference interval h is large.

The effect of varying m on parameter estimation was also studied. The value of m to be used depends upon the same factors as P_o, but has

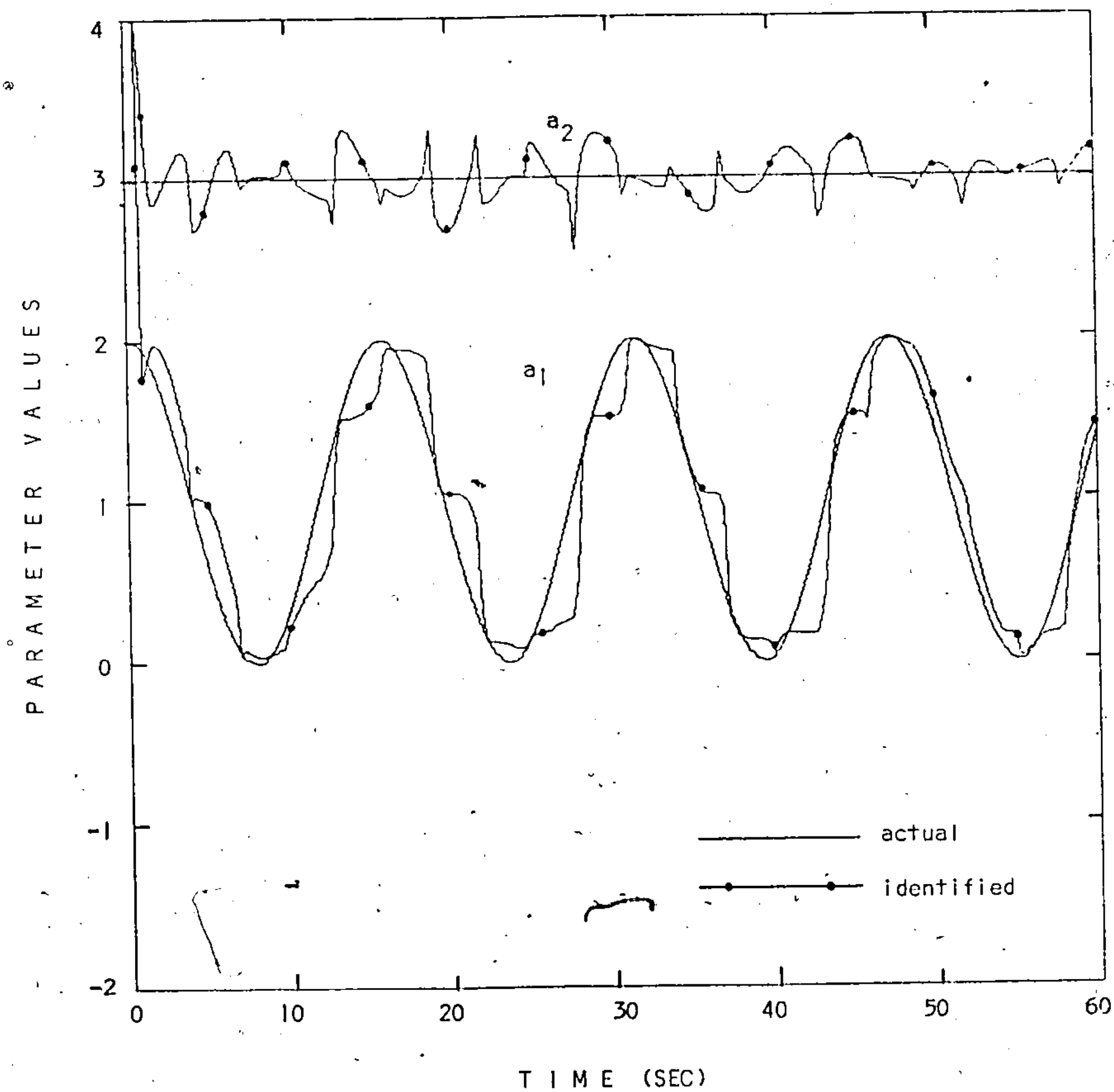

FIGURE 3: Parameter Tracking - Changing Covariance Matrix P Every 10 Iterations

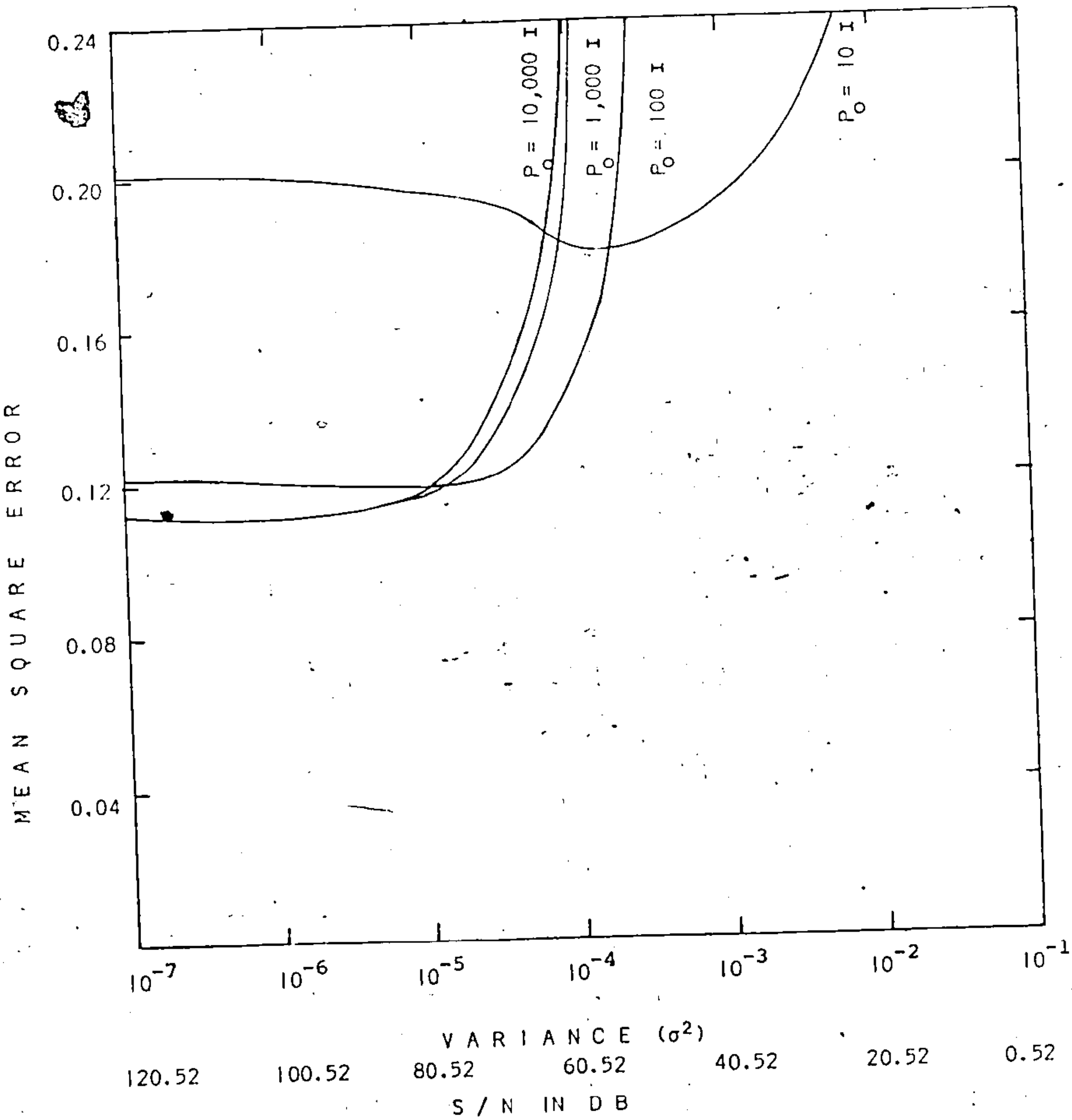

FIGURE 4: Effect of Noise Level on Nonstationary Parameter Estimation Assumed Piecewise Constant

the opposite effect. For example, when output data contains a higher level of noise, a lower value of P_o and a higher value of m was required.

(c) Parameter assumed to be varying randomly:

The system described by equation(5.1.1) was also identified by the procedure discussed in section 4.3.3, along with equation (4.1.1). The results shown in Figure 5, demonstrate rapid convergence with excellent tracking behaviour. As discussed before, in this procedure the covariance matrix P is modified by adding an arbitrary matrix C so as to give an artificial weight on the current data. It can be seen from equation (4.3.1) that C should be of the same dimension as the covariance matrix P, i.e. a n x n square matrix, where n is the order of the system. Different values for the elements of matrix C were used, in order to investigate the effect of C on identification. The best result was obtained when all elements of matrix C, except the i^{th} diagonal element were zero, where a_i is the time varying parameter. When more than one parameter is time varying, the corresponding diagonal elements should be non-zero and positive.

In the numerical example (5.1.1), the matrix C was taken as

$$C = \begin{bmatrix} c_{11} & 0 \\ 0 & 0 \end{bmatrix} .$$

For the results shown in Figure 5, a_{11} has a value of 100. The effect of varying c_{11} on parameter estimation is also considered. Figure 6 shows the mean squared error in the estimate of the nonstationary parameter (i.e. a_1) for different noise variances in the output using the values of c_{11} in the range of 1, 10 and 100. The values of c_{11} depends upon the same factors as P_o discussed above, i.e., rate of parameter

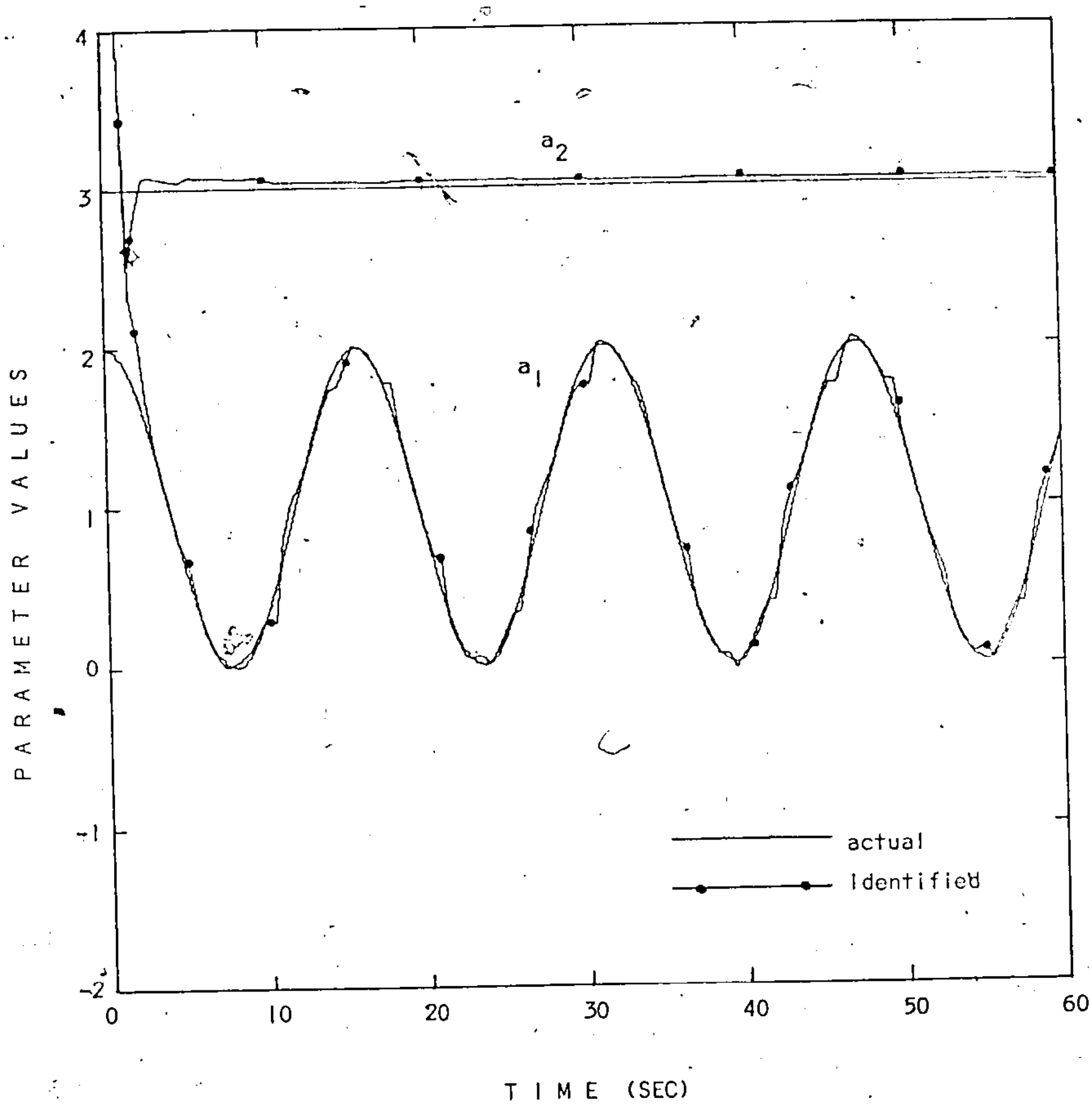

FIGURE 5: Parameter Tracking - Assuming Variation of Random Nature

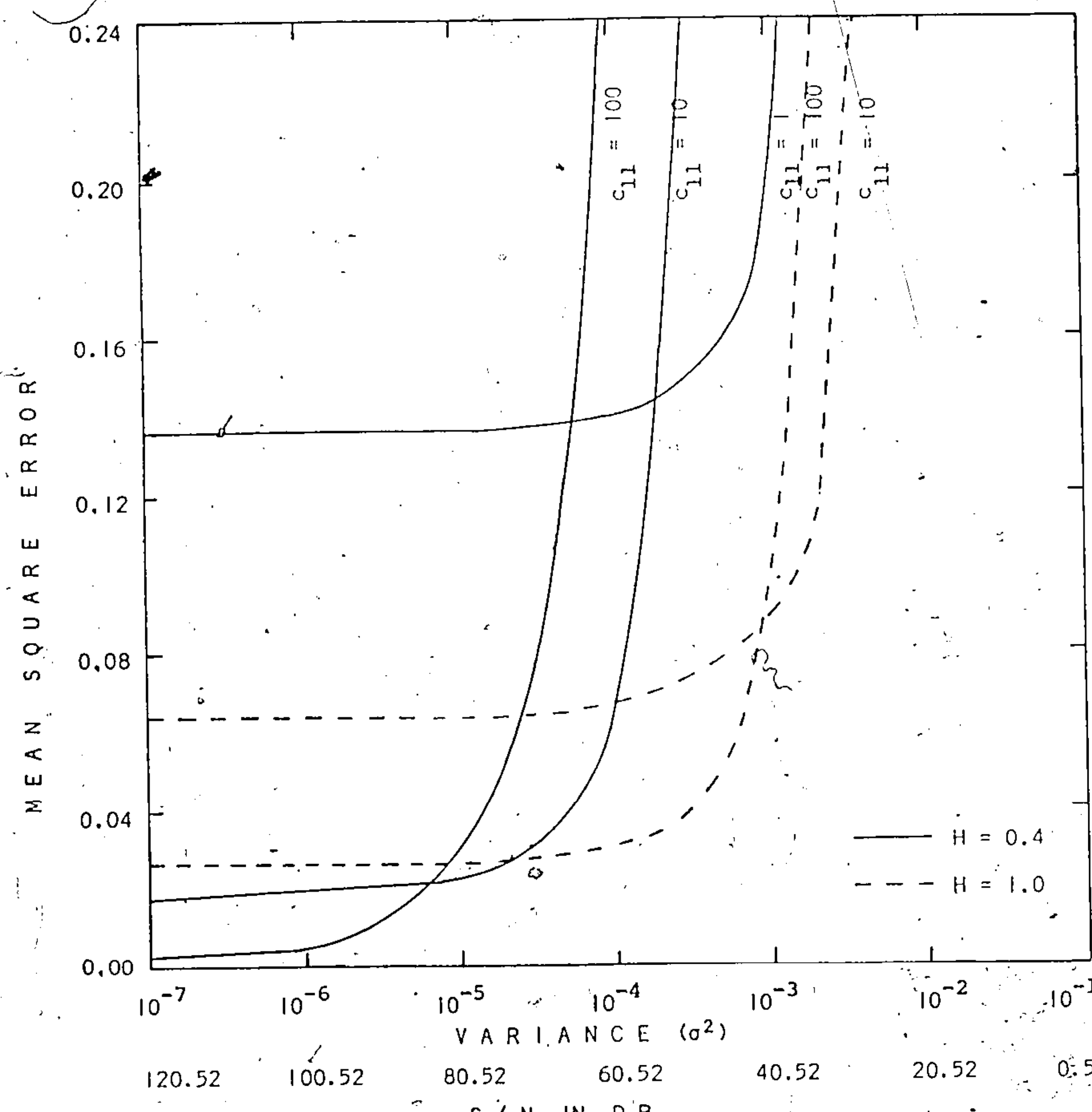

FIGURE 6: Effect of Noise Level on Nonstationary Parameter Estimation Assumed Varying Randomly

variation, presence of noise on output data, difference interval h, etc. and has the same effect as P_0. The non-zero diagonal elements of A should not be too large from noise point of view; at the same time it should be sufficiently large to follow the variation of parameter; a compromise between the two was needed.

Optimum value of a_{11} also depended upon the difference interval h. In Figure 6, the results using two different values of difference interval h(i.e. 0.4 and 1.0), are plotted. It is apparent from the figure that optimum value of a_{11} increases as difference interval h is increased. Using a value of h as 1, when a_{11} was taken as 1, the identification error was so large that the plot is shifted outside the range of the graph.

Comparison of three techniques:

The three techniques discussed for parameter estimation problem with unknown parameter variation are compared in order to determine the most efficient scheme. In Figure 7, the mean squared error is plotted as a function of the signal to noise ratio for the three methods. It is apparent from the figure that method 3, which assumed the parameter variation to be of random nature, is significantly more accurate than the other two methods. This is because the assumption of parameter variation of random nature is a more general and less restrictive assumption than the other two assumptions.

The technique 3 is also used for parameter estimation of another second order system, where parameter $a_1(t)$ is stationary and parameter $a_2(t)$ is varying with time. The system is described by the equation

$$\ddot{y} + a_1(t)\dot{y} + a_2(t)y = r(t) \qquad 5.1.4$$

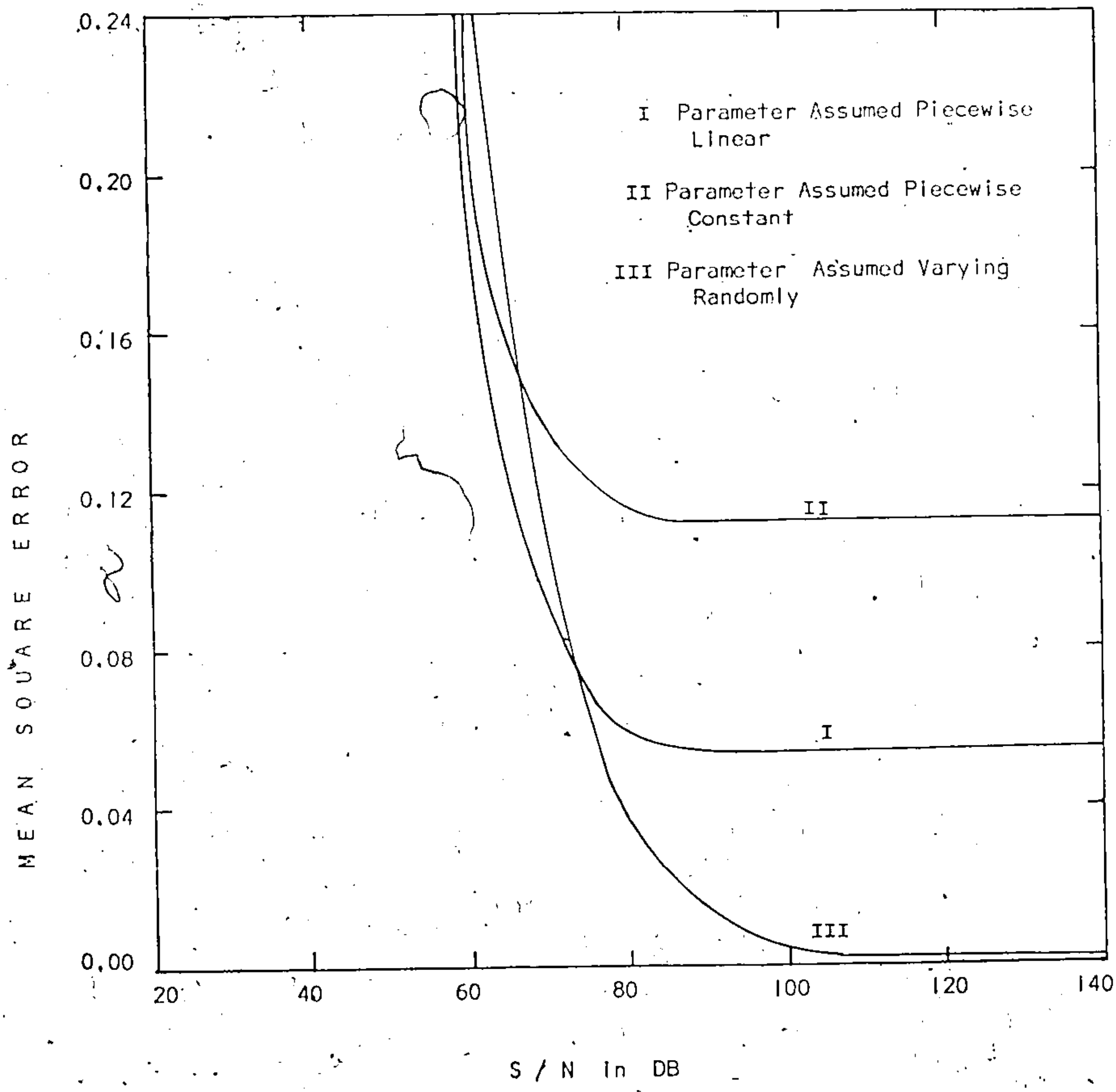

FIGURE 7: Comparison of the Three Method Discussed

where theoretically $a_1(t) = 4$, $a_2(t) = 2 + \sin(t)$ and $r(t) = \sin(t)$. The modifying matrix C is taken as

$$\begin{bmatrix} 0 & 0 \\ 0 & 100 \end{bmatrix} \qquad 5.1.5$$

Figure 8 shows the parameter tracking, indicating the effectiveness of the scheme. The scheme is also studied by adding wideband gaussian noise with the system output and the results are shown in Figure 9.

5.2. Effect of Varying Mesh Size

As observed before the value of difference interval h plays an important role in identification. The effect of h on identification was investigated for different noise levels so as to find some optimum value of h.

The system described by equation (5.1.1) was tested for different mesh sizes and the result is shown in Figure 10. For any particular noise level, there exists an optimal range of h outside of which the error grows up sharply. It can be observed from the figure that a larger value of h is preferred as noise level associated with the output increases. This occurs because the 'incremental signal to noise' ratio (in the sense of Balatoni [6]) which may be defined as

$$IS/N = \frac{\text{RMS Incremental Change in Signal}}{\text{RMS Incremental Change in Noise}} \qquad 5.2.1$$

$$\simeq \sqrt{\frac{\frac{1}{N}\sum_{i=0}^{N}(y_i - y_{i-1})^2}{\sigma^2}} \qquad 5.2.2$$

where y_i is the noise free output signal,

σ^2 is the noise variance

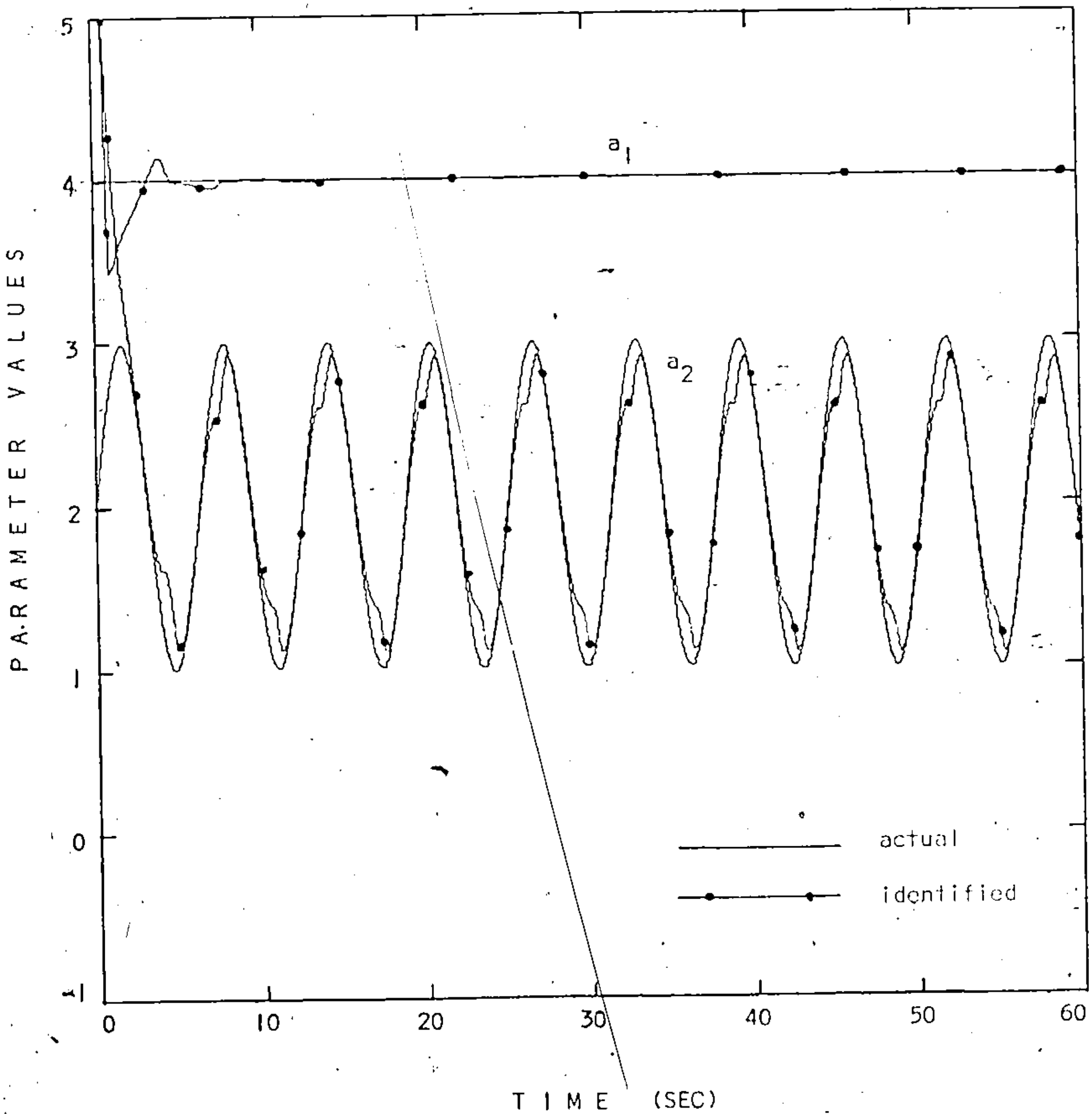

FIGURE 8: Parameter Tracking of a Rapidly Varying System

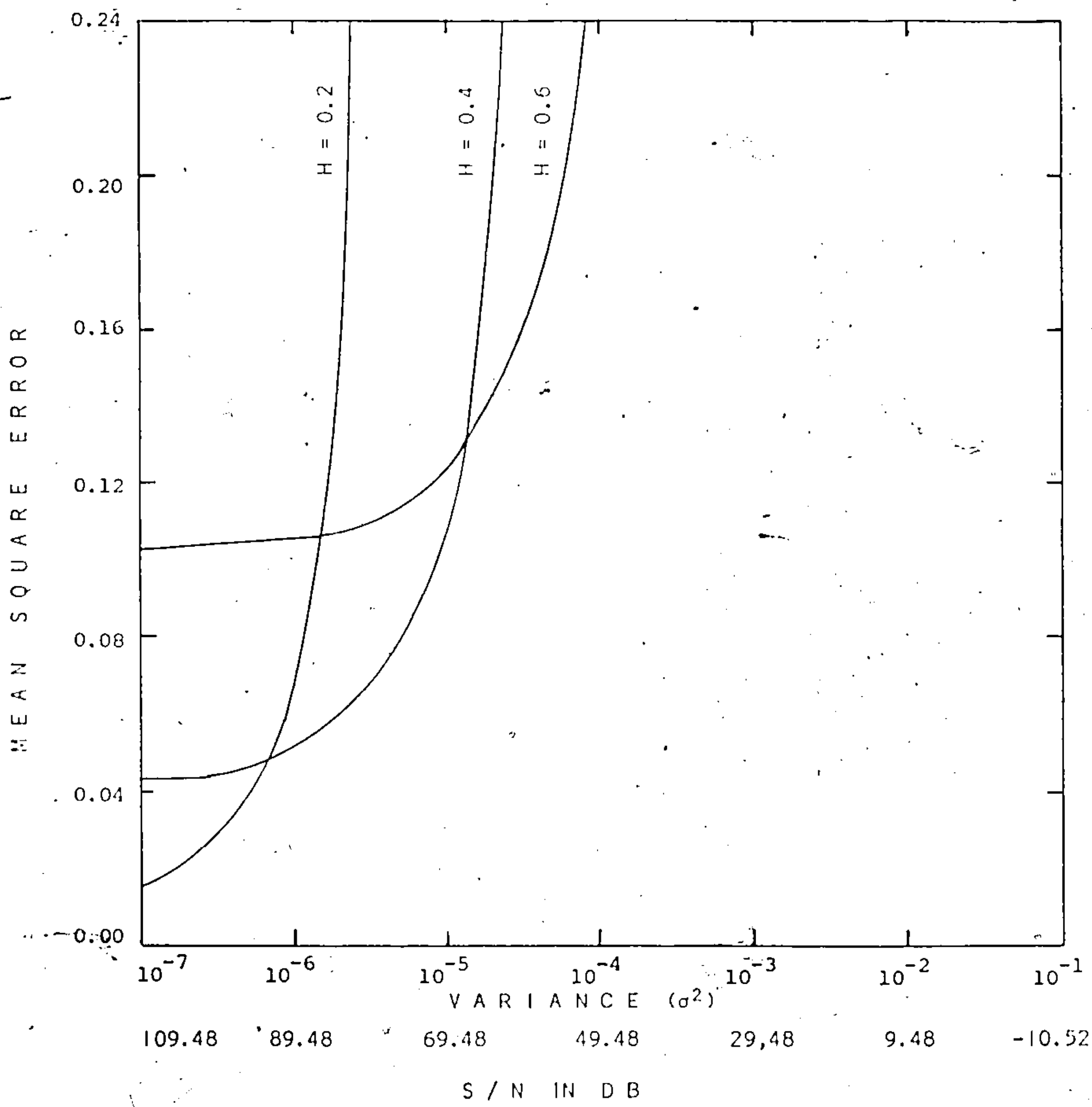

FIGURE 9: Effect of Noise Levels on Time Varying Parameter Identification

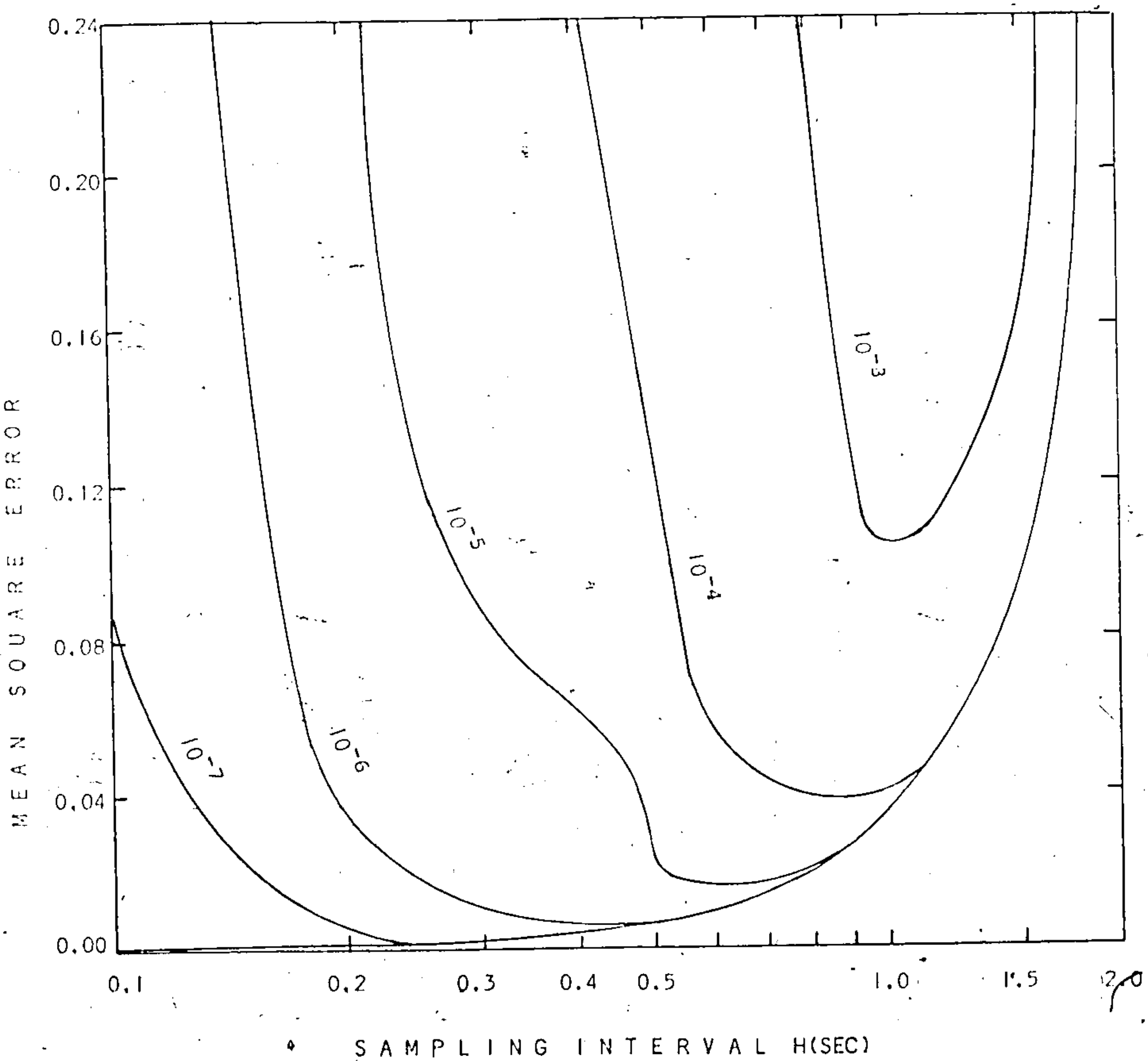

FIGURE 10: Effect of Varying Sampling Interval For the Time Varying Parameter

decreases with increase of difference interval h. This behaviour also can be explained from the sampling theorem point of view. The output of a system excited by deterministic continuous periodic signal is approximately band limited, whereas the noise term is uniformly distributed in the frequency domain and is wideband in nature. Consequently, when the noise corrupted output is sampled with higher values of h, the samples carries less information about the noise compared to the signal and the identification is more accurate. However, when the sampling interval is too large, sampling theorem is violated in sampling the output itself. In Figure 10, the maximum value of h for which the identification is acceptable is about 1.0 sec. which is slightly less than that permitted by sampling theorem.

To get a measure of effectiveness of identification for time varying parameter, let us assume that the parameter consisted of a constant component and a time varying component so that the parameter a_i can be written as

$$a_i = a_{ic} + a_{iv} \qquad 5.2.1$$

where a_{ic} = constant component; and $a_{iv} = a_i - a_{ic}$, is the time varying component. Let us define error coefficient ε_c and ε_v, namely

$$\varepsilon_c = \left| \frac{a_{ic} - a'_{ic}}{a_{ic}} \right| \times 100 \qquad 5.2.2$$

$$\varepsilon_v = \frac{\sum_{i=1}^{N} (a_{iv} - a'_{iv})^2}{\sum_{i=1}^{N} (a_{iv})^2} \times 100$$

$$= \frac{\text{mean square error in time varying part}}{\text{mean square value of the time varying part}} \times 100 \qquad 5.2.3$$

where $a_i^!$ is the identified parameter value with

$$a_{ic}^! = \text{the constant part and } a_{iv}^! = a_i^! - a_{ic}^! \tag{5.2.4}$$

For example, for the system described by the equation (5.1.1)

$$a_{1c} = 1 \tag{5.2.5}$$

and $\quad a_{1v} = \cos(0.4t)$

so that mean squared value of a_{1v} is 1/2.

The accuracy of identification required, actually depends upon the particular system one deals with. However, for our analysis, it was assumed that the identification will be acceptable for values of both ε_c and ε_v as high as 10. It is seen that ε_c is comparatively less than ε_v. Even when ε_v has a value as high as 10, ε_c has value in the range of 1. So that, one can write

$$\varepsilon_v \cong \frac{\text{mean square error}}{\text{mean square } a_{iv}} = \frac{\frac{1}{N}\sum\left(a_i - a_i^!\right)^2}{\text{mean square } a_{iv}} \tag{5.2.6}$$

The overall results gives the guideline for optimal mesh interval for accurate parameter estimation as

1) IS/N > 20
2) S/N > 60 db
3) sampling theorem is not violated. 5.2.7

The second condition, however, is not affected by the size of the mesh interval.

5.3. Comparison With Central Difference Method

As the spline identification technique looks quite similar to other finite difference methods, identical systems were also identified by the central difference method. The main difference between the central difference method and the spline method can be observed by comparing the mean squared error when h and the noise variance σ are varied. In Figure 11, the two methods are compared for the system described by equation (5.1.1), with three noise variances 0, 10^{-6} and 10^{-4}. The dotted line corresponds to the central difference method and the solid line is due to spline identification. Results of the spline are seen to be superior in the sense that these work for higher value of h, and higher value of h is desirable from noise point of view. This occurs because the continuity conditions of first and second derivatives at the mesh points which effect a smoothing of the derivatives, permit the identification algorithm to be used at a higher value of h; no such continuity conditions are considered in the case of central difference methods.

5.4. System Implementation on Analog Computer

To test the practical feasibility of the identification scheme discussed for linear time varying systems, a few systems were implemented on an EAI 580 analog/hybrid computer; the output was sampled using an approximate 8-bit accuracy A/D converter of the AX08 Laboratory peripheral and the PDP-8/I digital computer. The block diagram of the sampling procedure is shown in Figure 12. These samples were then used to identify the systems on IBM 360/65 digital computer.

The system implemented on the analog computer is given by

$$\ddot{y} + a_1(t)\dot{y} + a_2(t)y = r(t) \qquad 5.4.1$$

where $a_1(t) = 1$; $a_2(t) = 3 + \sin(0.1t)$ and $r(t) = \sin(t)$.

The sampling was done in noisy environment. The A/D conversion was not accurately linear. Figure 13 shows the actual and identified parameters.

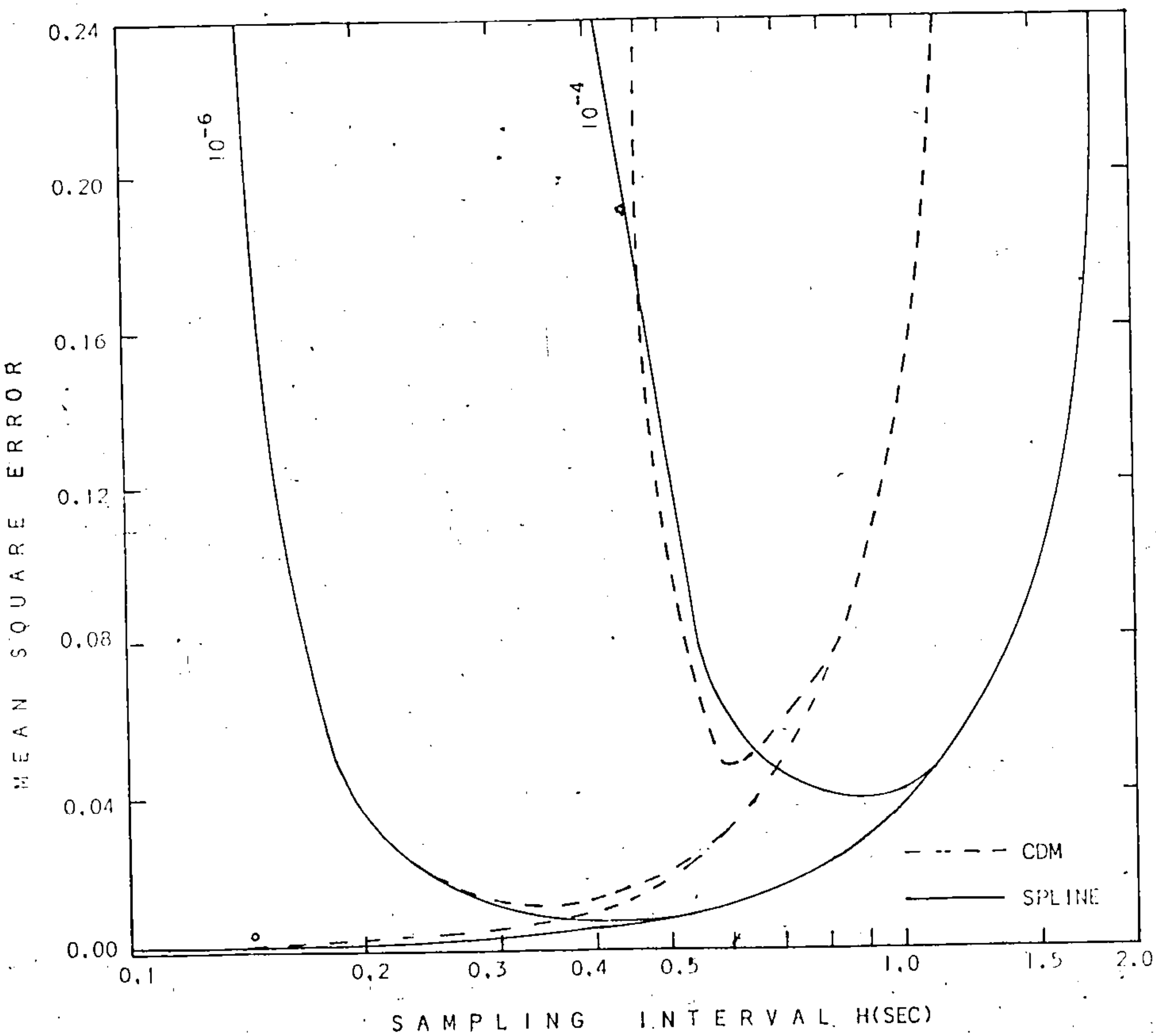

FIGURE 11: Mean Square Error in Time Varying Parameters vs Sampling Interval for Spline and Central Difference Method

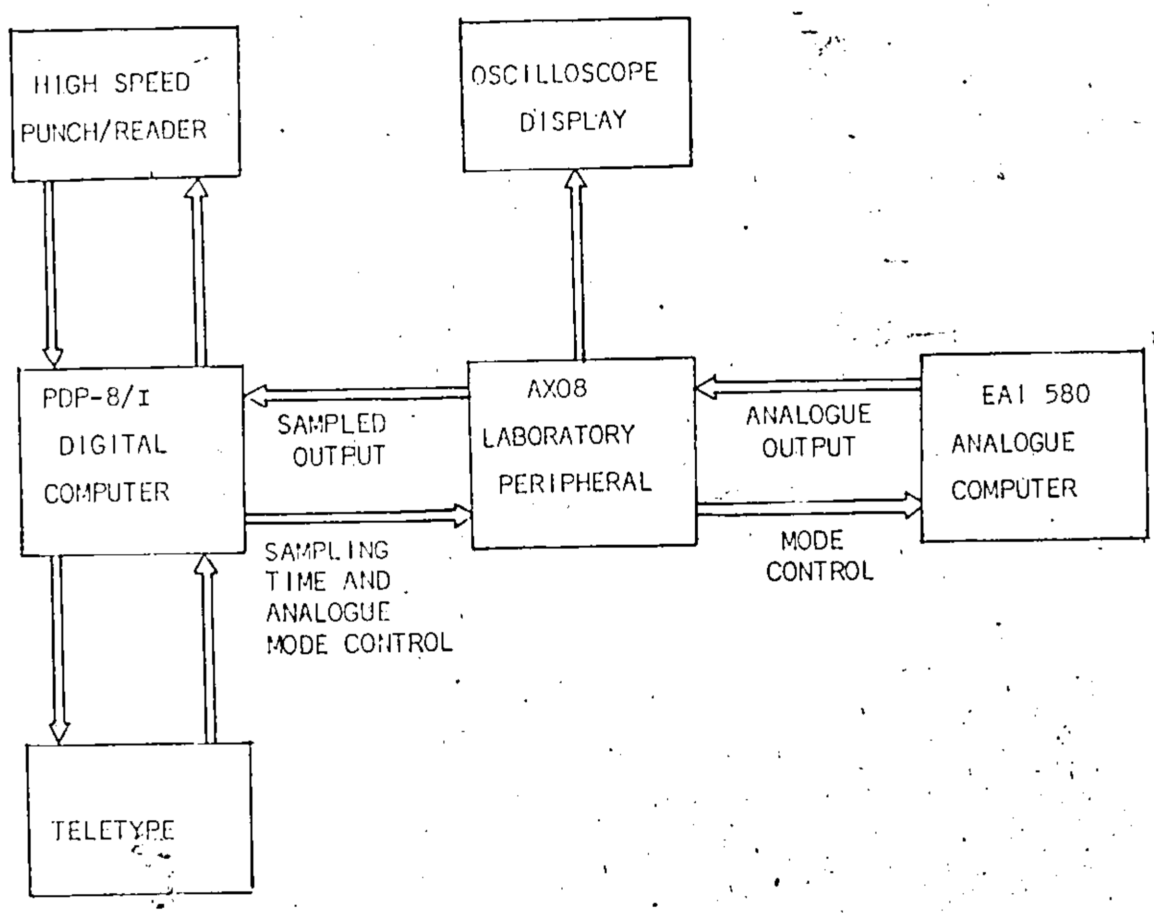

FIGURE 12: Block Diagram For Sampling Physical System Outputs

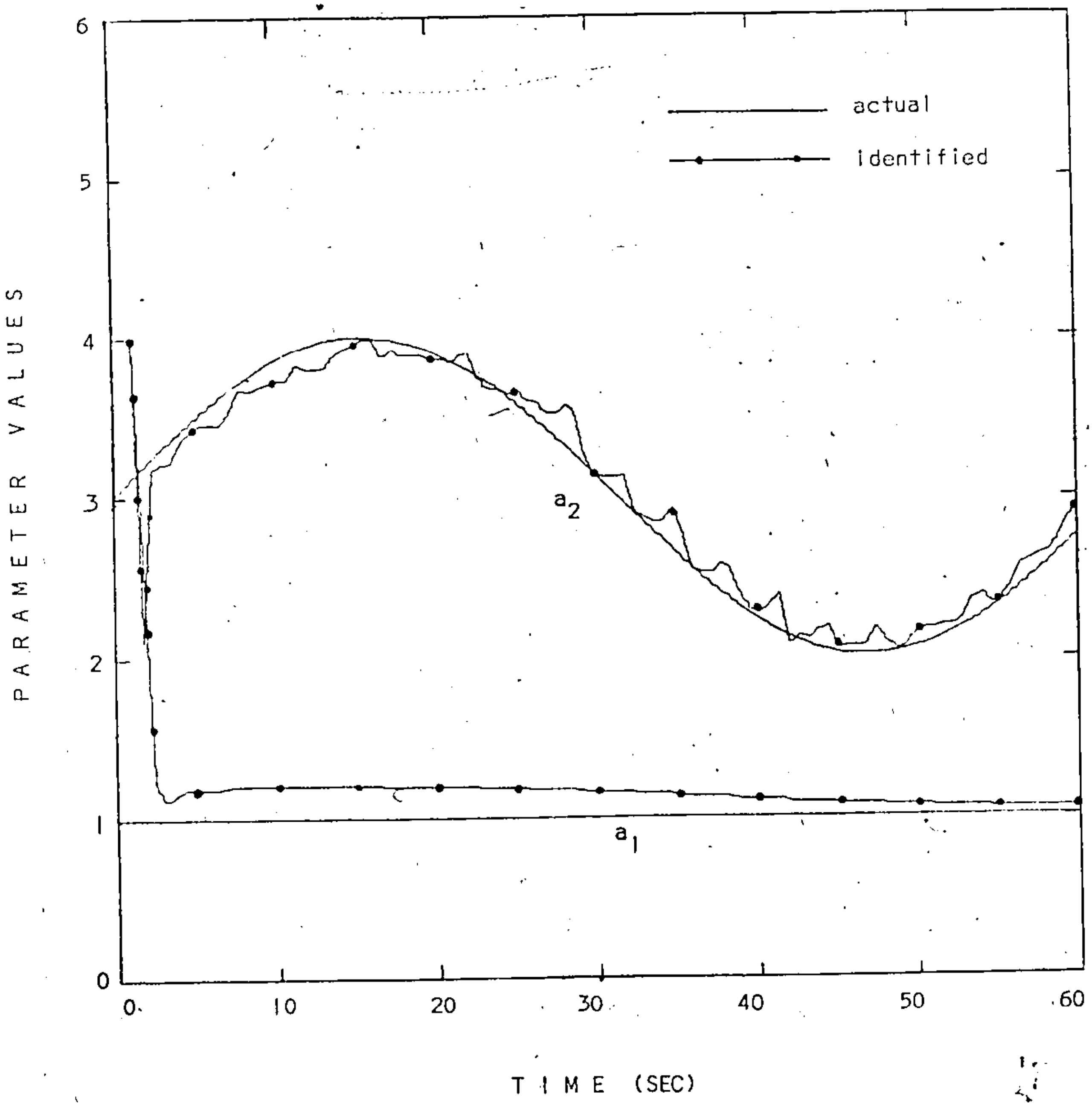

FIGURE 13: Parameter Tracking of System Simulated on EAI 580 Analogue Computer

The results demonstrate that the identification exhibits satisfactory tracking behaviour.

The same system was also identified using step, triangular and rectangular inputs, and it was found that parameter estimations were more accurate when the system was excited by a sinusoidal input. Since the sinusoidal input, along with its derivatives is smooth and continuous, the resulting estimates were more accurate.

Figure 14 and 15 demonstrate the tracking behaviour of two other second order systems implemented on the analog computer. Both of them were excited by sinusoidal inputs.

5.5. Identification of Nonlinear Systems

Identification scheme developed in the last chapter for nonlinear systems was tested for a few numerical examples on the IBM 360/65 digital computer. In the initial stage of the work, the system output was obtained by solving the equation by Adams-Moulton prediction correction formula [24] on the digital computer.

In the first example, the system to be identified is given by

$$\ddot{y} + a_1\dot{y}^2 + a_2 y = r(t) \qquad 5.5.1$$

where theoretically $a_1 = 2$, $a_2 = 7$ and $r(t) = 2 \sin(t)$. Initially a_1 and a_2 were assumed as 6 and 8 respectively, when the technique developed in section 4.6 was applied along with recursive leastsquares method. The parameters as shown in Figure 16 converged rapidly to their theoretical values, showing the effectiveness of the technique in parameter estimation. The behaviour of the algorithm was also observed when the output of the system was corrupted by white gaussian noise in the variance range: 10^{-7}, 10^{-6}, ..., 10^{-1}. Figure

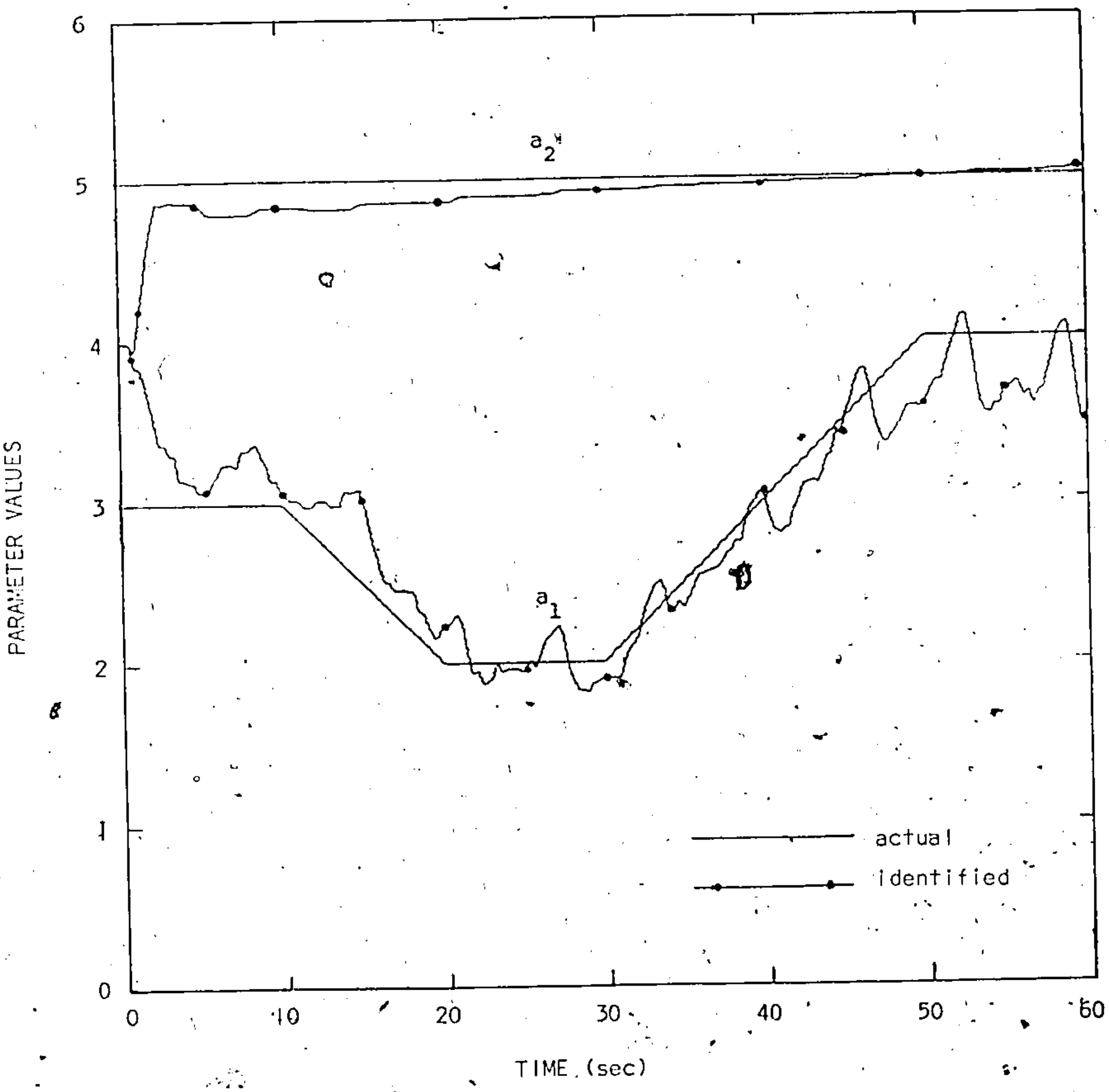

FIGURE 14: Parameter Tracking of an Analogue Simulated System.

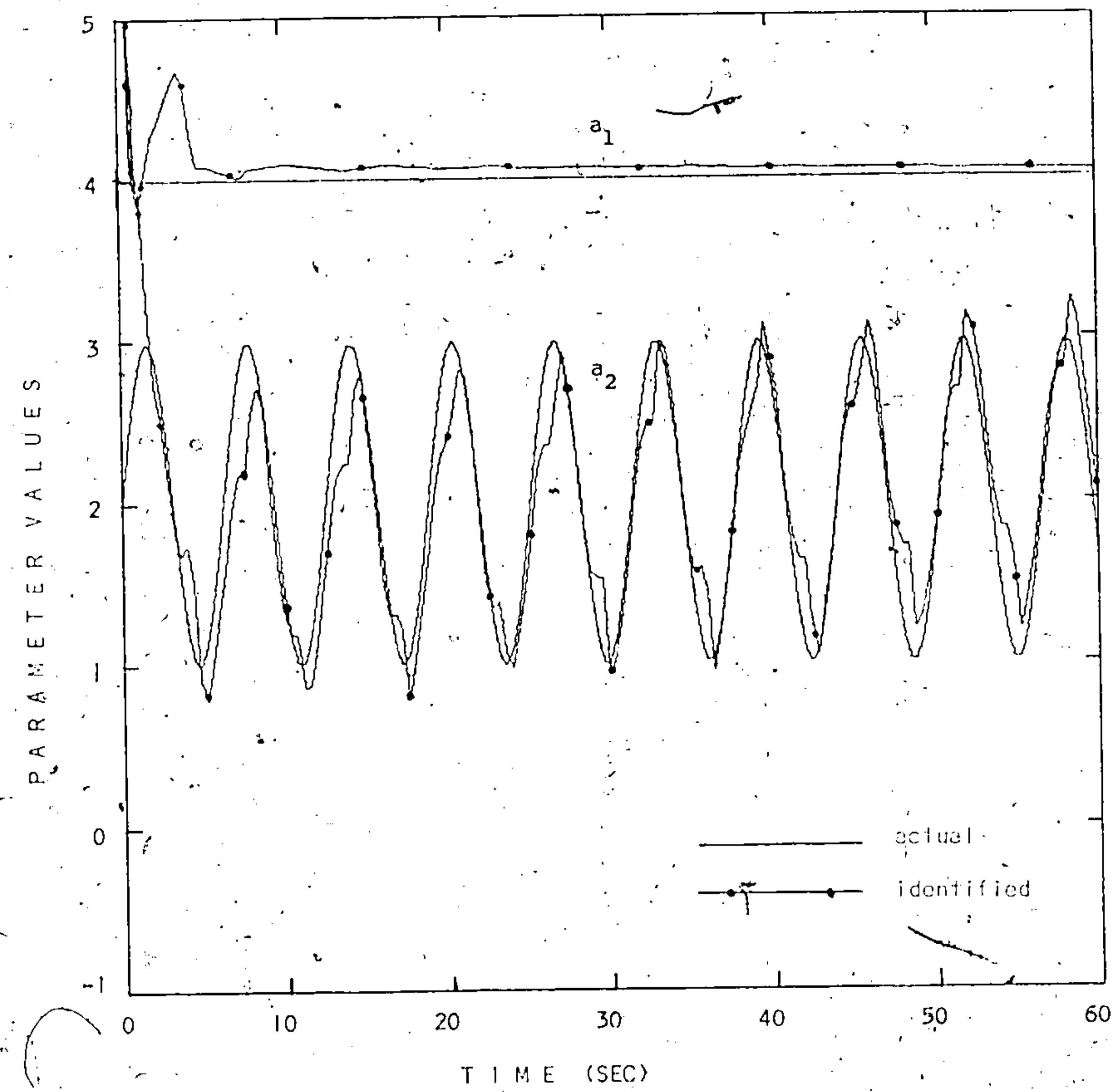

FIGURE 15: Identification of Analogue Simulated System

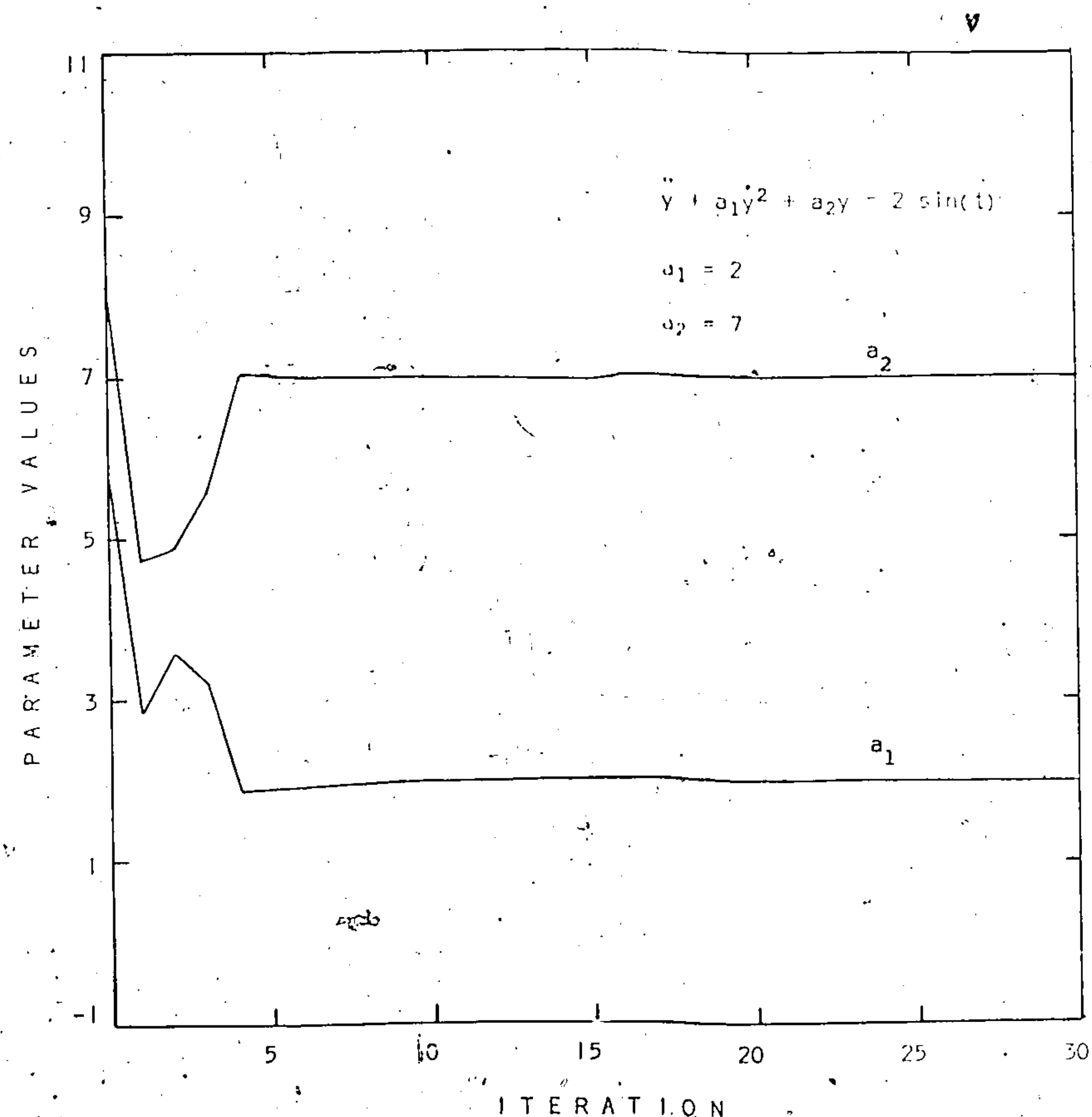

FIGURE 16: Rate of Convergence of Identified Process Parameters of Second Order Nonlinear System

17 shows the result in terms of |% error| against signal to noise ratio. h was taken as 0.3. The results demonstrate that if a 10% error in parameter estimation is allowable, then the identification scheme is acceptable up to about 40 db signal to noise ratio.

In another example, the system was given by the nonlinear equation

$$\ddot{y} + a_1\dot{y}y + a_2y = r(t) \qquad 5.5.2$$

with theoretically $a_1 = 1$, $a_2 = 4$ and $r(t) = \sin(t)$.

Spline derivative generation on a uniform mesh as discussed in section 4.5 along with the recursive least squares algorithm was used in the identification. The parameters converged rapidly to their actual values. The percentage error in parameter estimation with additive noise in the output is shown in Figure 18.

As discussed in section 4.6, for a particular class of non-linear second order systems given by the equation

$$\ddot{y} + a_1\dot{y} + a_2y^s = r(t) , \qquad 5.5.3$$

application of cubic spline results in the closed form identifying equation (4.6.2). This equation along with recursive least squares algorithm was applied to the system described by equation (5.5.3), with $a_1 = 9$, $a_2 = 1$ and $s = 2$. The resulting identification, as shown by Figure 19, demonstrates the feasibility of the technique.

5.6. Effect of Varying Mesh Size

The system described by equation (5.5.1) was identified using a range of sampling intervals, in order to ivestigate the effect of sampling interval in spline identification. In Figure 20 and 21, the percentage error in identification of the parameters a_1 and a_2 respectively are plotted with respect to the mesh size for different

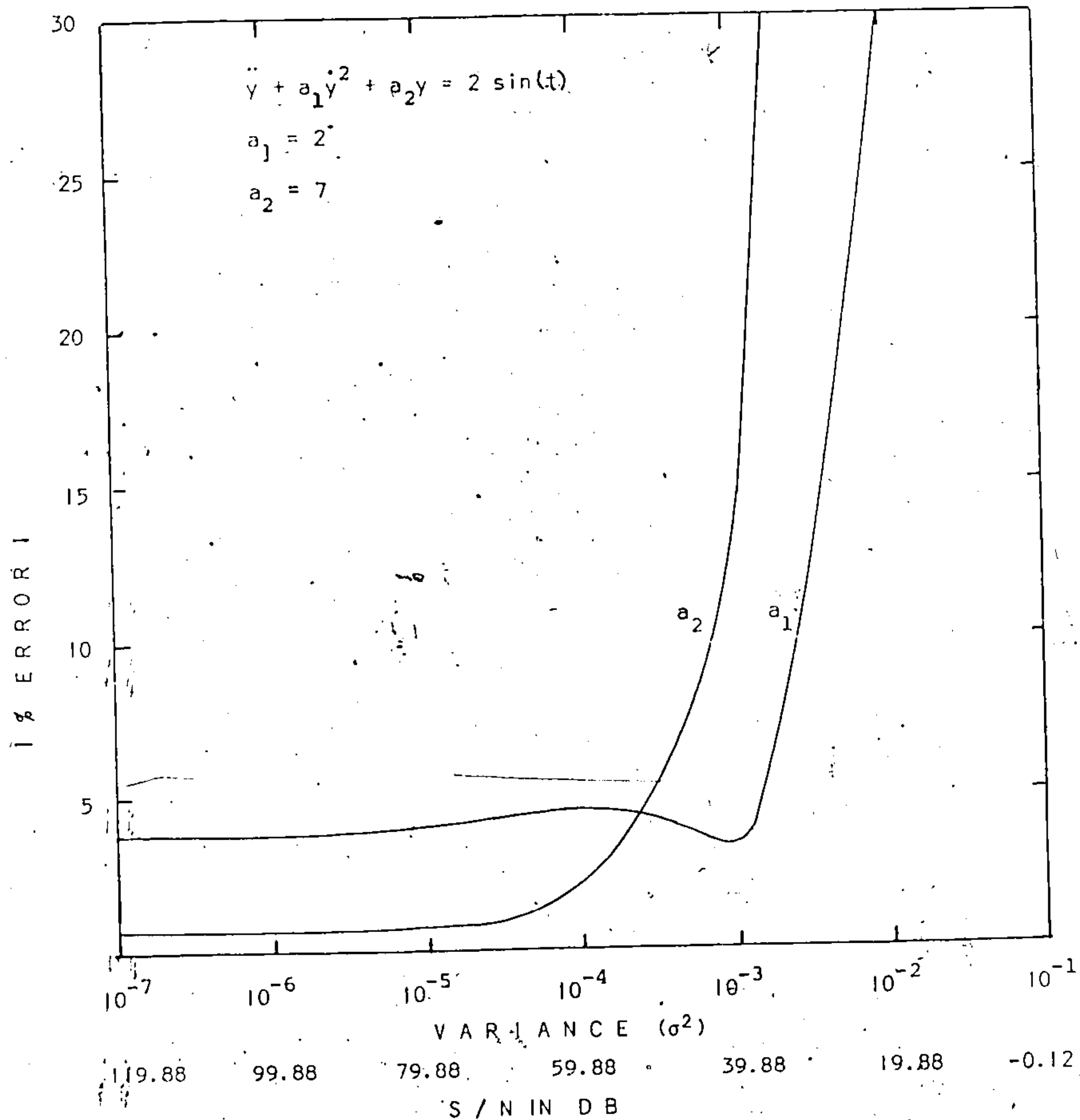

FIGURE 17: Effect of Noise on Parameter Estimation of Second Order Nonlinear System

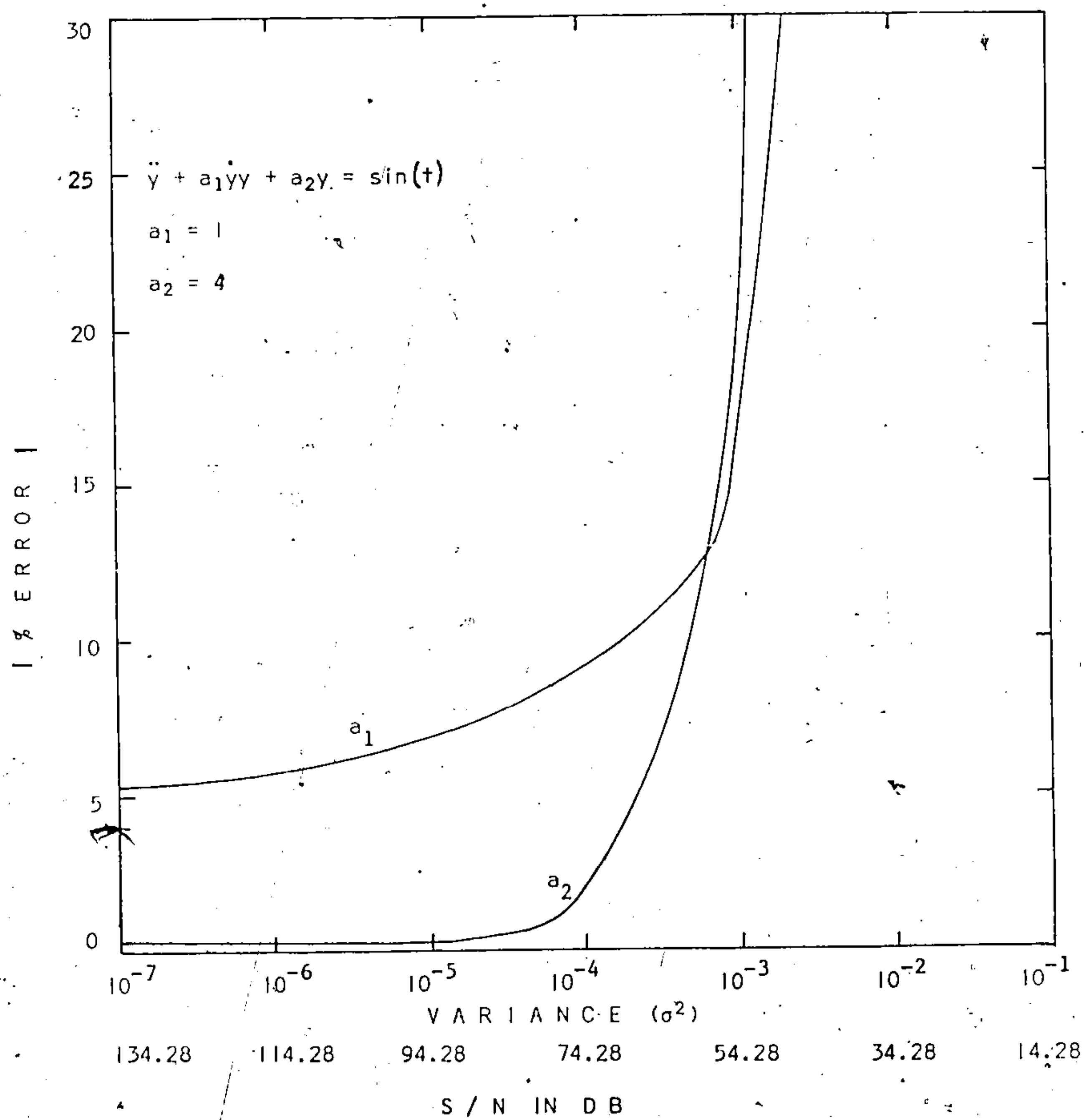

FIGURE 18: Effect of Noise on Identification of Second Order Nonlinear System

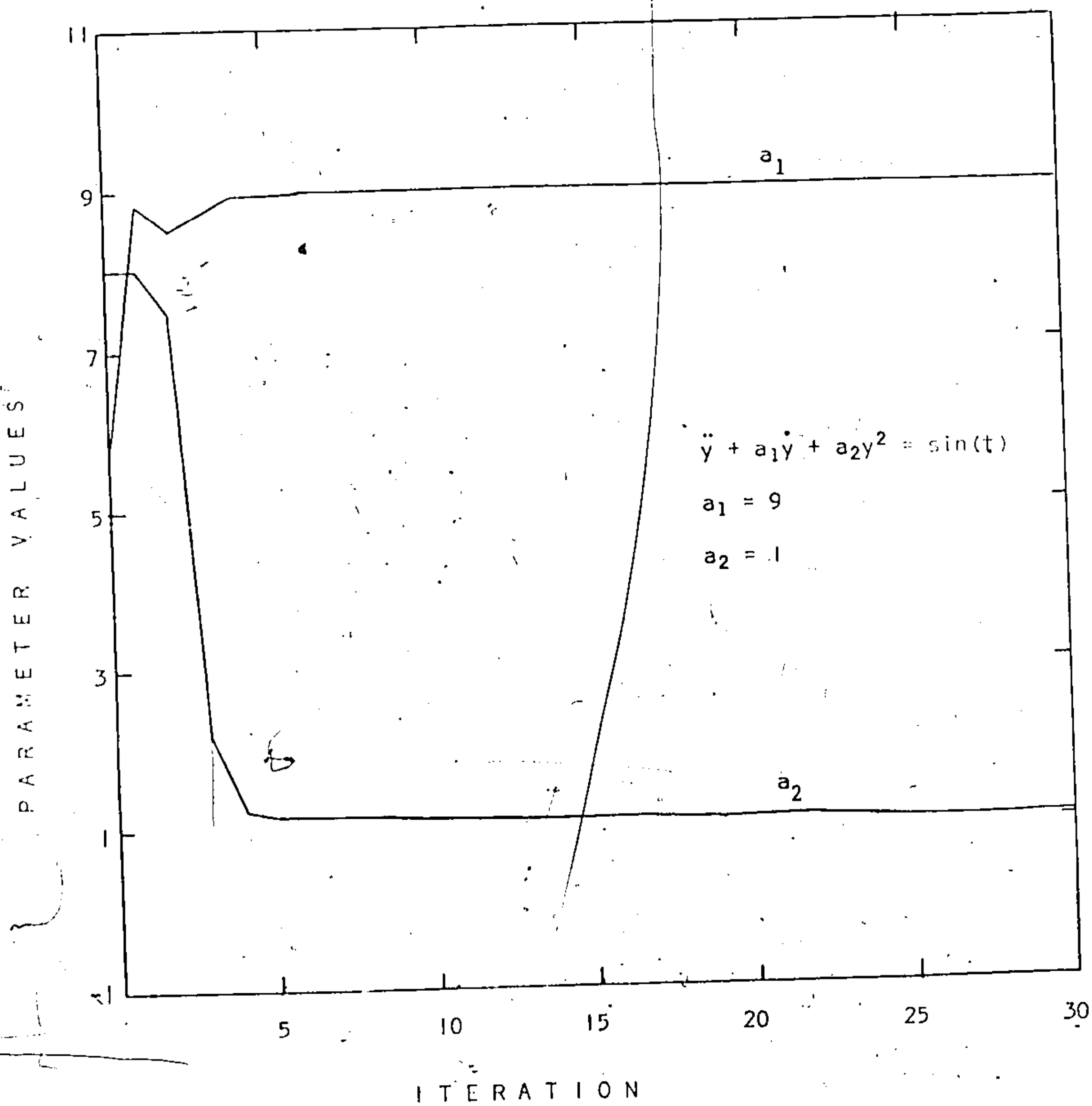

FIGURE 19: Parameter Convergence of Second Order System

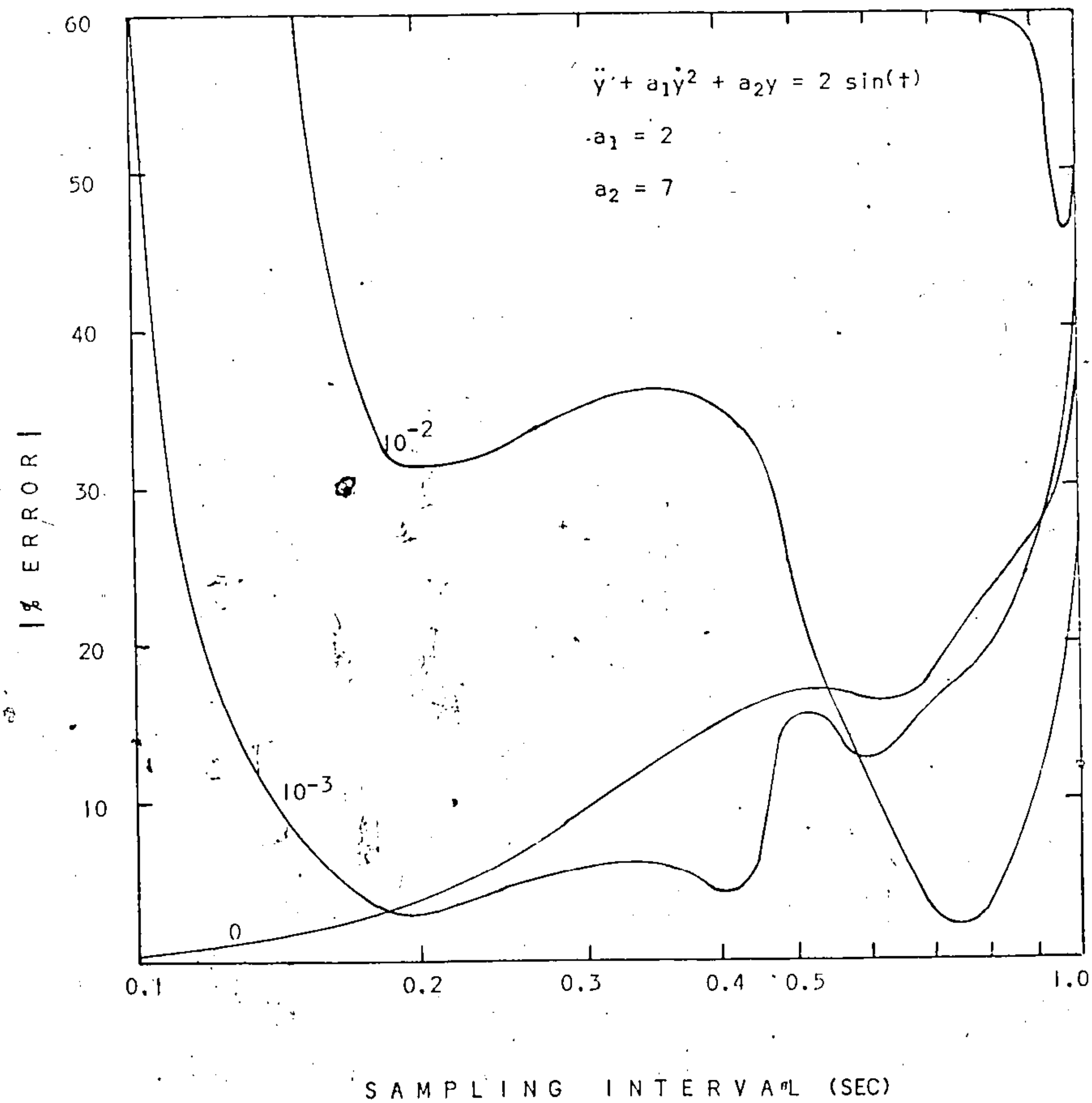

FIGURE 20: Family of | % Error | Curves For Parameter a_1 Showing the Effect of Variation of Sampling Interval Size

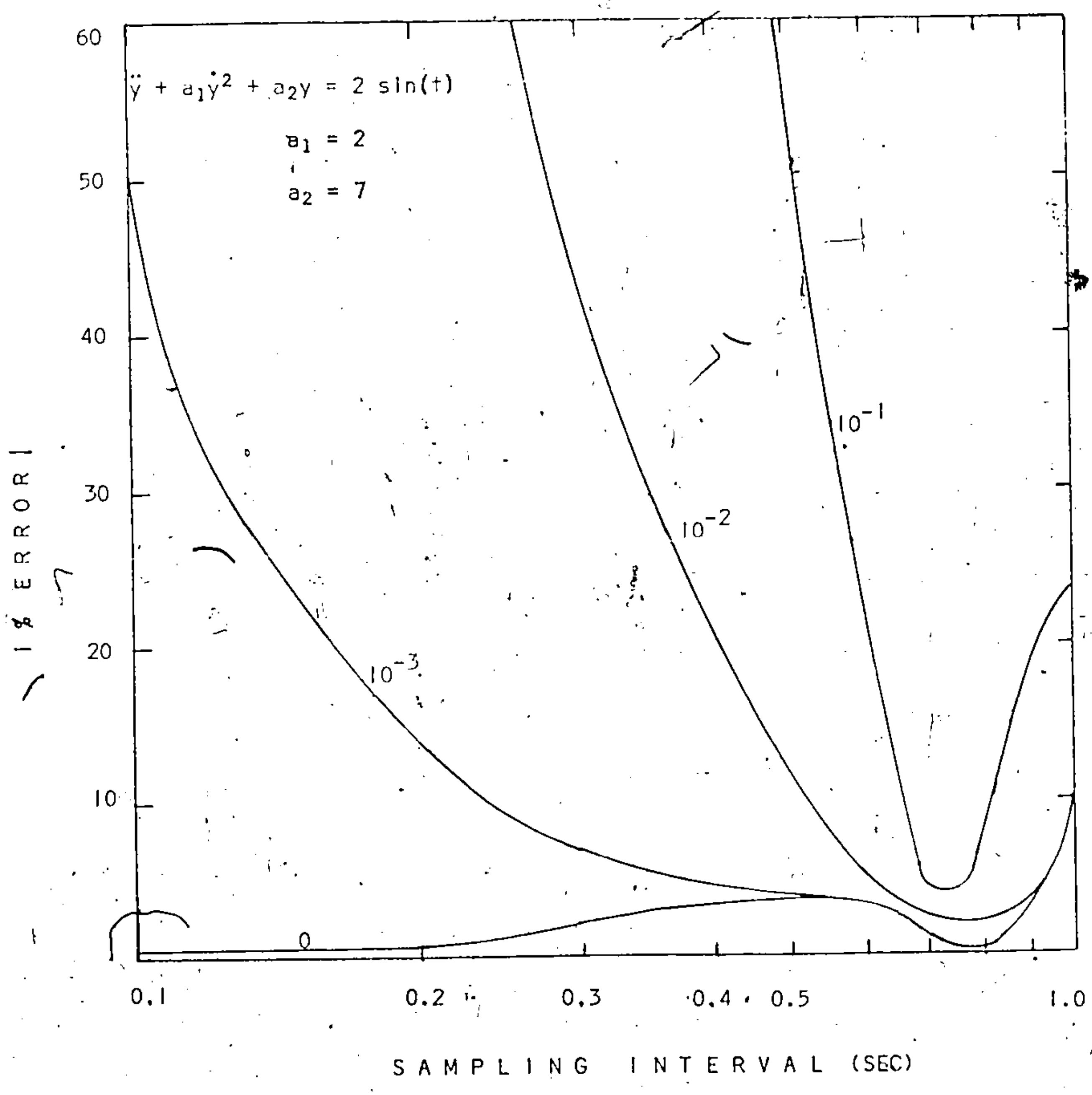

FIGURE 21: Family of |% Error| Curves For Parameter a_2 Showing the Effect of Variation of Sampling Interval Size

range of noise variance. As before, we note here that though at small noise variances the error increases with mesh size, when output is contaminated by higher levels of noise, a larger value of mesh size is preferable. The reasoning advanced in section 5.2 for similar behaviour by a linear system applies here.

5.7. State Variable Approach

In section 4.7, the identification procedure for systems described by a set of first order differential equation was applied in the system given by

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + r_1(t)$$
$$\dot{x}_2 = a_{21}x_1 = a_{22}x_1x_2 + r_2(t) \qquad 5.7.1$$

where theoretically $a_{11} = 0$, $a_{12} = 1$, $a_{21} = 4$, $a_{22} = 1$, $r_1(t) = 0$ and $r_2(t) = \sin(t)$. Basically the system is the same as the system described by equation (5.5.2), with $x_1 = y$ and $x_2 = \dot{y}$. When both x_1 and x_2, i.e., all the state variables, are available as data, the parameters a_{11}, a_{12}, a_{21}, and a_{22} were identified accurately with rapid convergence. The effect on identification when both the state variables were disturbed with random noise of the same variance was also investigated and the results are shown ~~in Figure 22.~~

When all state variables are not available as data, the problem can be tackled only if some of the parameters are known. For example, if the values of a_{11} and a_{12} are known, and writing $a_{21} = a_1$, $a_{22} = a_2$ one obtains

$$y = x_1$$
$$\dot{x}_1 = x_2 \qquad 5.7.2$$
$$\dot{x}_2 = a_2 x_2 + a_1 x_1 x_2 + r(t)$$

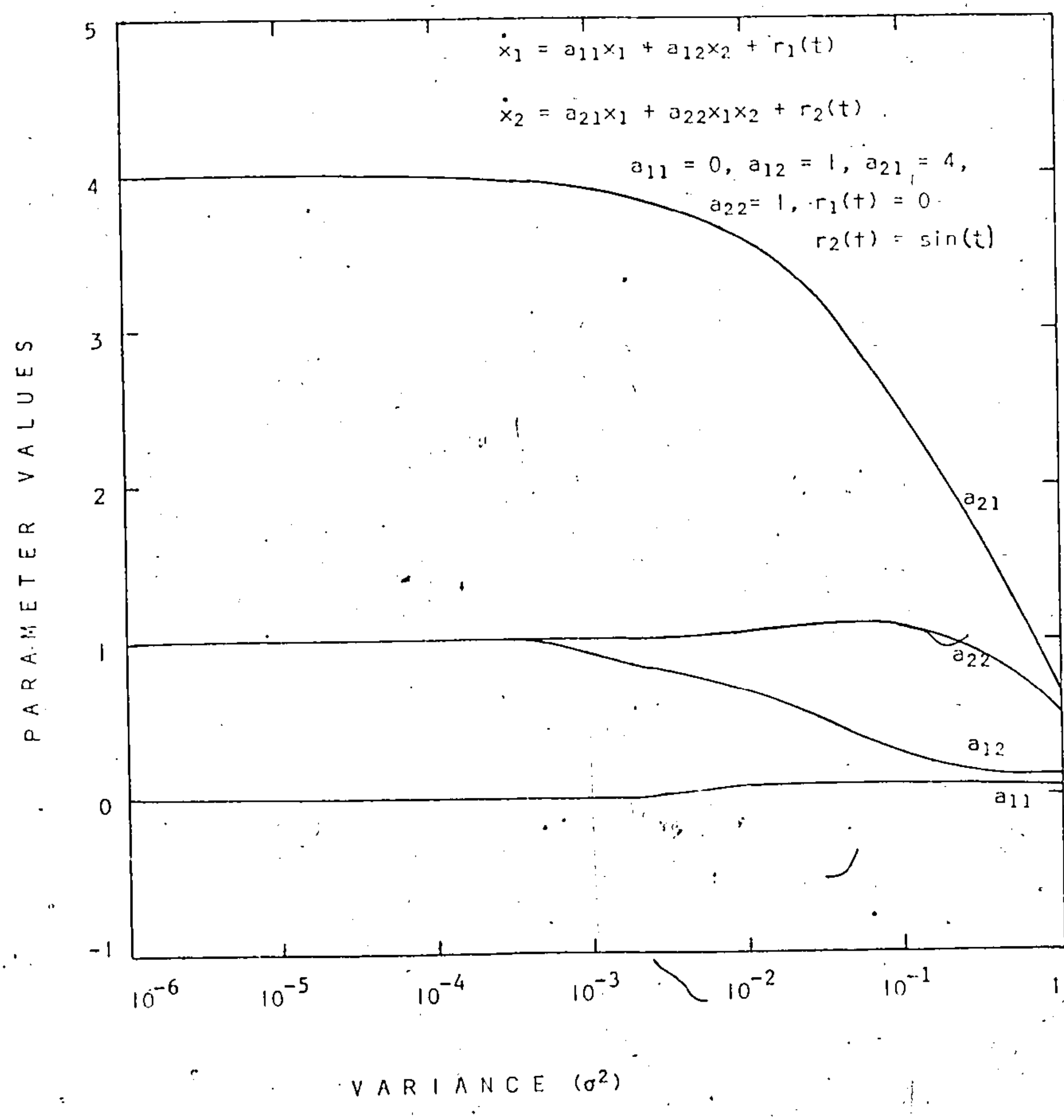

FIGURE 22: Effect of Noise on Identification of Multi-Dimensional System Described by a Set of First Order Differential Equations

where $a_1 = 1$, $a_2 = 4$, $r(t) = \sin(t)$ and y is the observation. The algorithm discussed in section 4.7 for estimating x_2 was used. Even though the rate of parameter convergence was slower, however, a higher noise level was tolerated. Figure 23 shows that the identification is acceptable up to a noise variance of about 3×10^{-3}, which corresponds to 40db signal to noise ratio. The value of h used was 0.3.

The system described by equation (5.5.1) was also identified using the same technique. In state equations (5.5.1) can be written as

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = a_2 x_2 + a_1 x_1^2 + r(t) \qquad 5.7.3$$

with $a_1 = 2$, $a_2 = 7$ and $r(t) = 2\sin(t)$. The result is presented in Figure 24 as percentage error for different levels of additive noise in the output.

The effect of varying mesh size on identification using the state-variable-approach was also investigated. The results are presented in Figure 25 and Figure 26, as percentage error in parameter estimation against mesh size, for different levels of noise. The plots look rather messy, however the general trend is that high levels of noise restrict the acceptable values of h to a narrower range, and the maximum allowable value of h is slightly less than that permitted by the sampling theorem. Assuming 10% error in identification is permitted, the guide-line for optimum sampling interval for nonlinear systems can be given as

(1) IS/N > 5.0

(2) S/N > 40 db

(3) Sampling theorem is not violated.

Spline identification of single input/output system using state equations

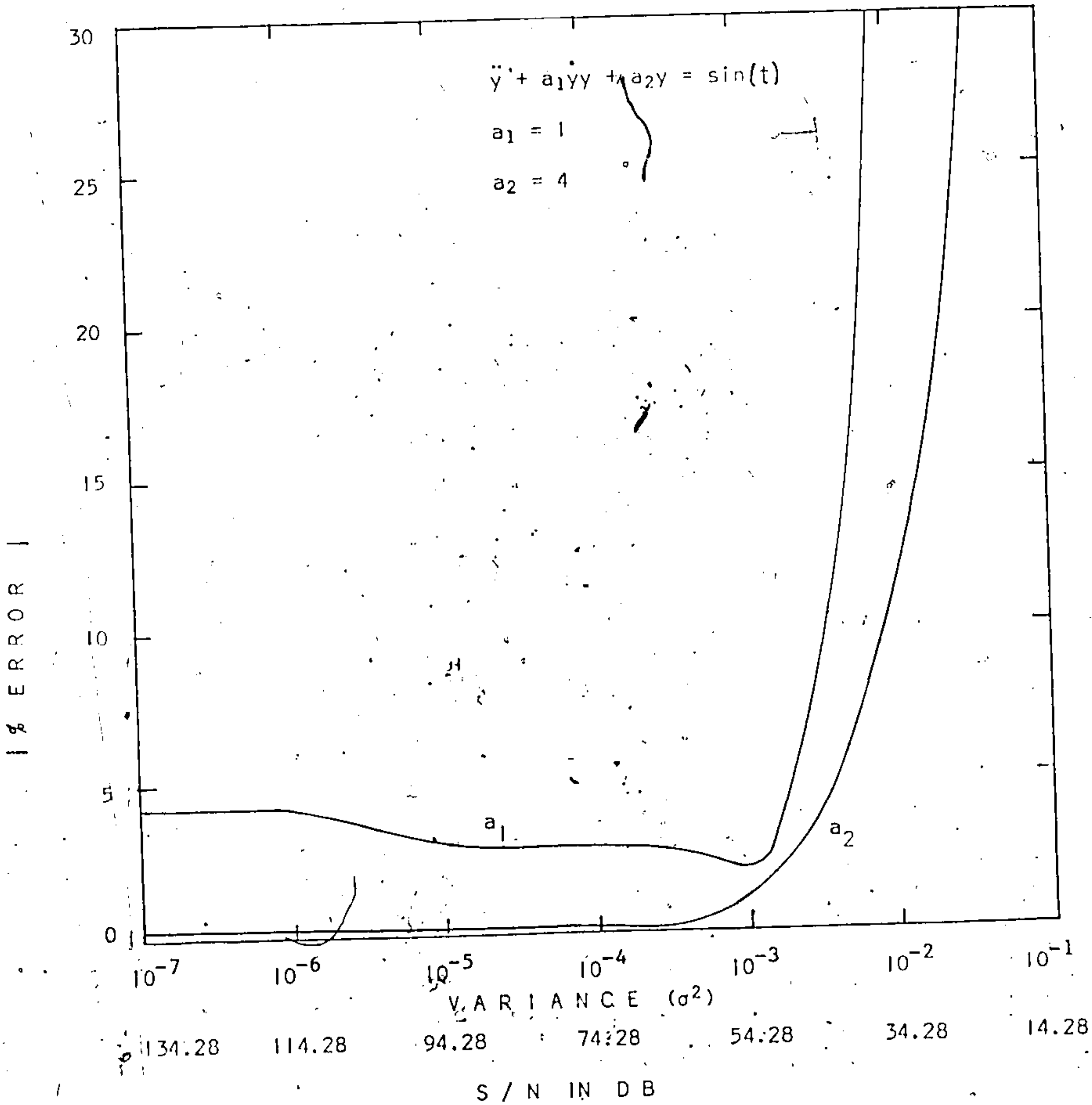

FIGURE 23: Effect of Noise on Parameter Estimation by State Variable Approach

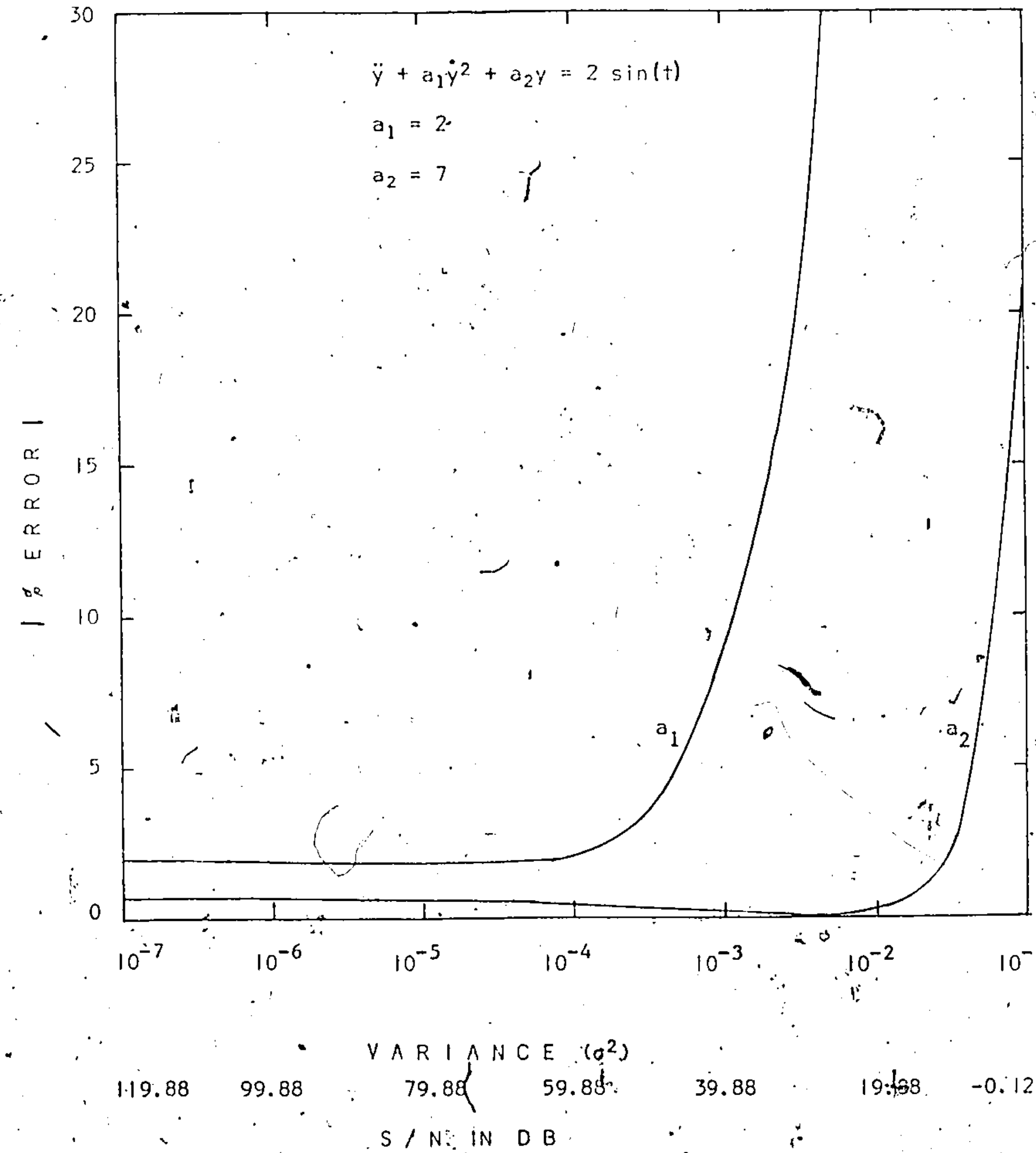

FIGURE 24: Effect of Noise on Parameter Estimation by State Variable Approach

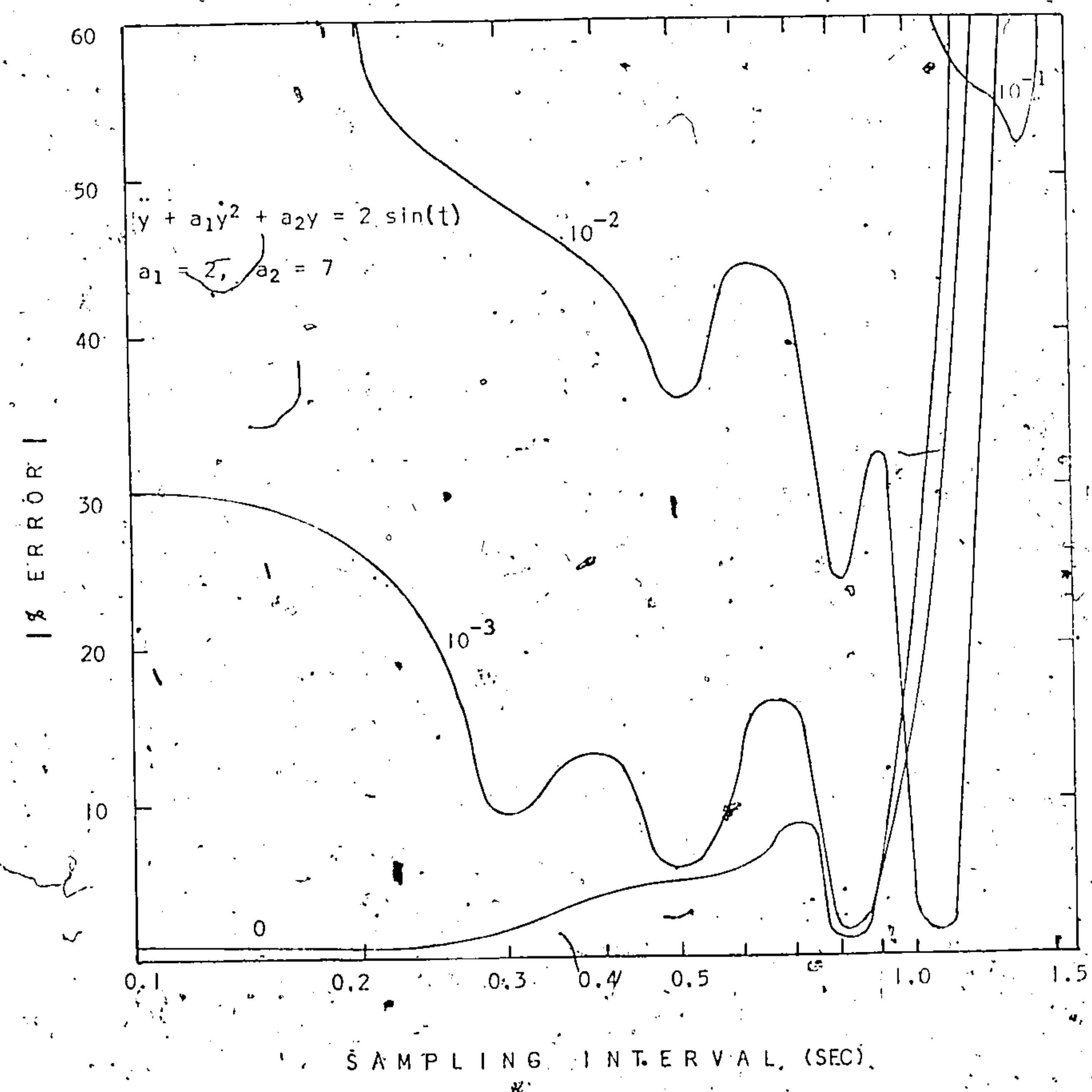

FIGURE 25: Effect of Varying Sampling Interval in Estimation of Parameter a_1 by State Variable Approach

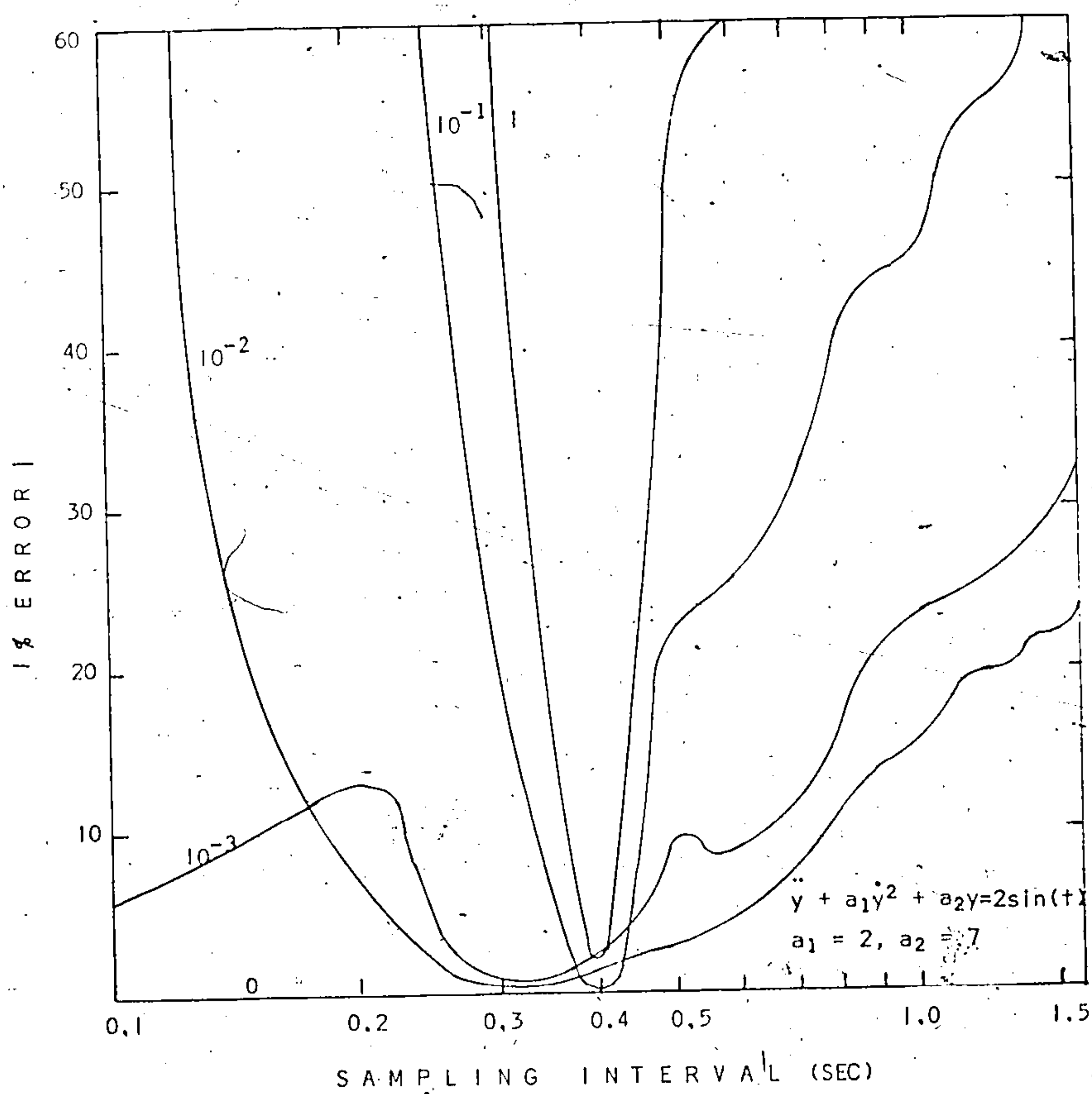

FIGURE 26: Effect of Varying Sampling Interval in Estimation Parameter a_2 by State Variable Approach

actually corresponds to spline on spline fit of derivatives of the output. As a result, the derivatives are more smooth, hence suitable when the output is only approximate. However, the disadvantage is the off-line estimation and also the procedure involves a matrix inversion of the same dimension as the number of data points. The inversion algorithm for this particular type of matrix discussed in section 2.7, however, cannot be used when the number of data points is more than 125, due to computer overflow problem. If computer time can be sacrificed, the matrix inversion can be done by the conventional matrix inversion procedure, and the algorithm can be extended to cover large number of data points.

5.8. Comparison With Central Difference Method

The system described by equation (5.5.1) was also identified by derivative generation with the central difference method along with recursive least squares algorithm. As a comparison, the parameter estimation by the central difference method, spline method and spline on spline (state variable approach) is plotted as percentage error in a_1 and a_2 for different "signal to noise ratio" in Figure 27 and Figure 28 respectively. Difference interval h was 0.3 sec. for all of them. Results of spline technique were shown to be superior over central difference method as faster convergence and higher accuracy was obtained. Result of spline on spline technique was shown to be the best among them as it was more accurate at higher levels of noise. The only disadvantage is that it is off-line in nature. Therefore, when on-line estimation is not necessary, the state variable technique can be applied to obtain more accurate parameter estimates.

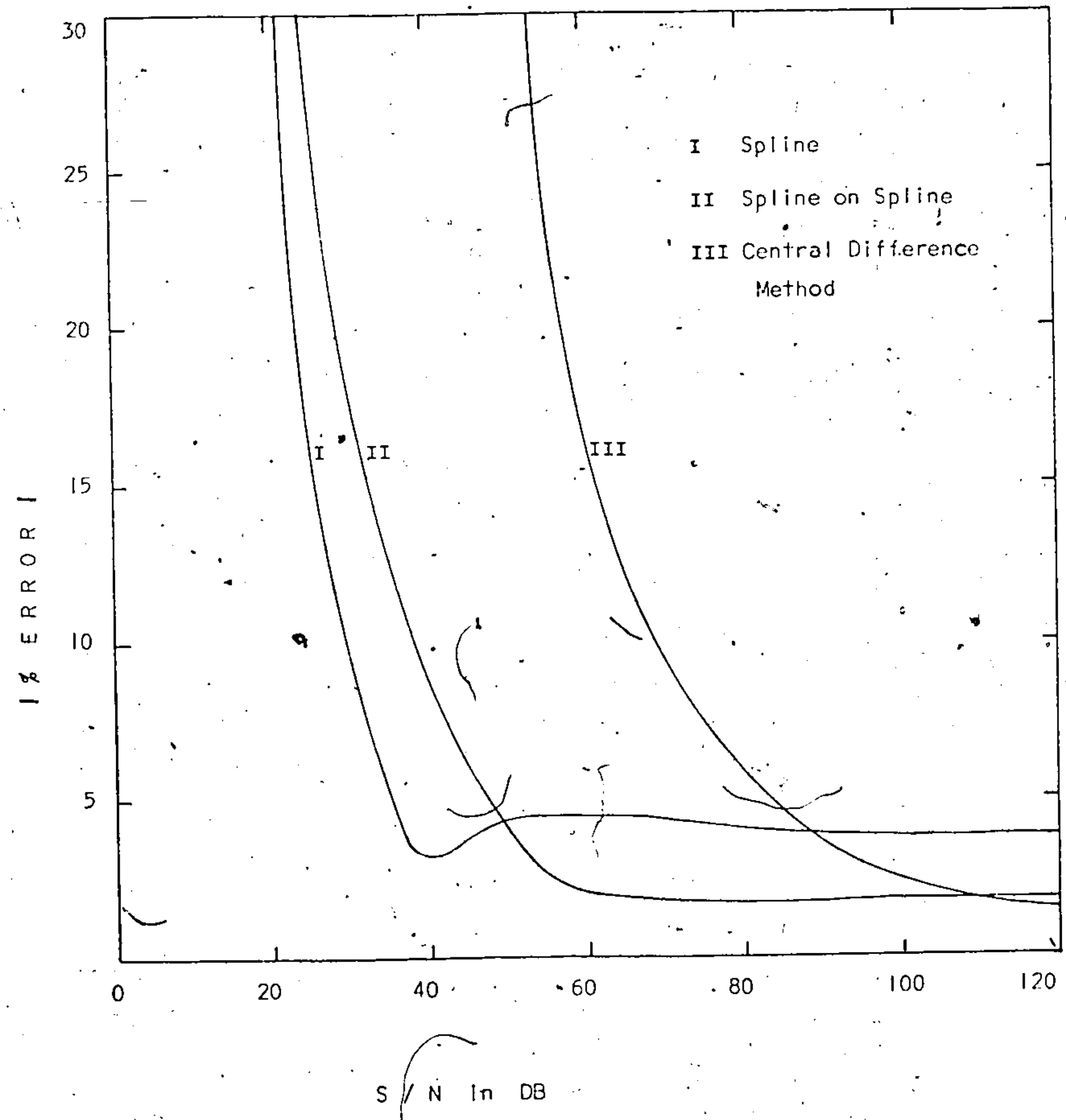

FIGURE 27: Comparison of the Three Methods In Estimating Parameter a_1 For Different Signal to Noise Ratio

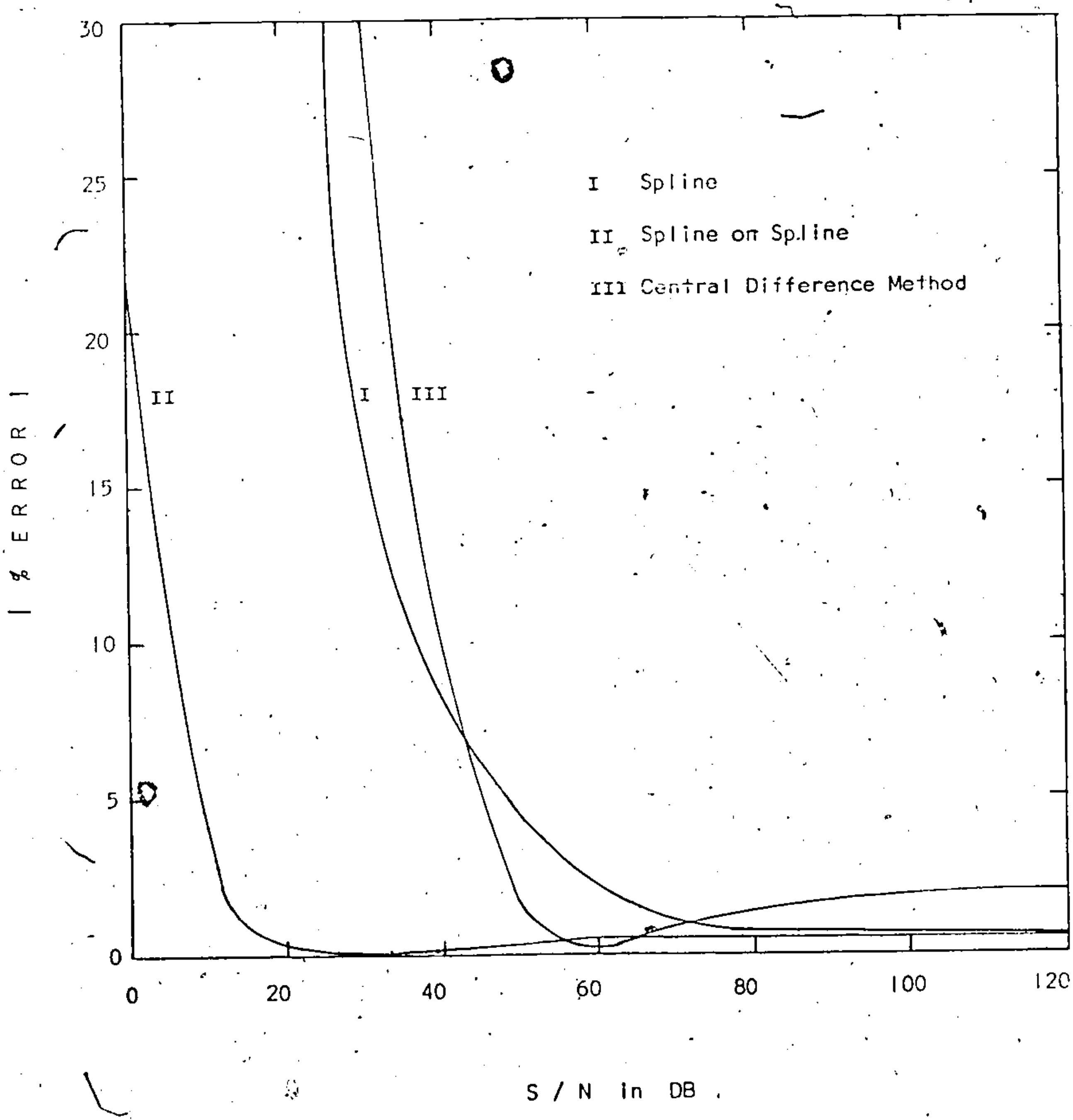

FIGURE 28: Comparison of the Three Methods in Estimating Parameter a_2 for Different Signal to Noise Ratio

The same systems were also implemented on an EAI 580 analog/hybrid computer and sampled as shown in the block diagram of Figure 12. The A/D conversion was not accurately linear. This data was then tested by the said techniques to observe the parameter estimation in a practical noisy system.

The parameter estimates for the system given by equation (5.5.2) was seen to converge rapidly as shown in Figure 29 indicating the usefulness of splines in system identification.

The system given by equation (5.5.1) was also implemented and the resulting estimates converged to the theoretical parameter values only in 14 iterations. The effect of noisy measurements on the identification is shown in Figure 30. The identification is acceptable to about 50 db additive noise with these data. When state variable technique is used as an off-line estimation, as shown in Figure 31, the identification is acceptable with about 40 db additive noise.

Other types of excitation functions such as rectangular waves and triangular waves were also used. Figure 32 shows the convergence of parameters using spline and state variable approach when the system described by equation (5.5.2) is excited by rectangular wave in the analog computer. However, when noise was considered, sinusoidal input yielded better results.

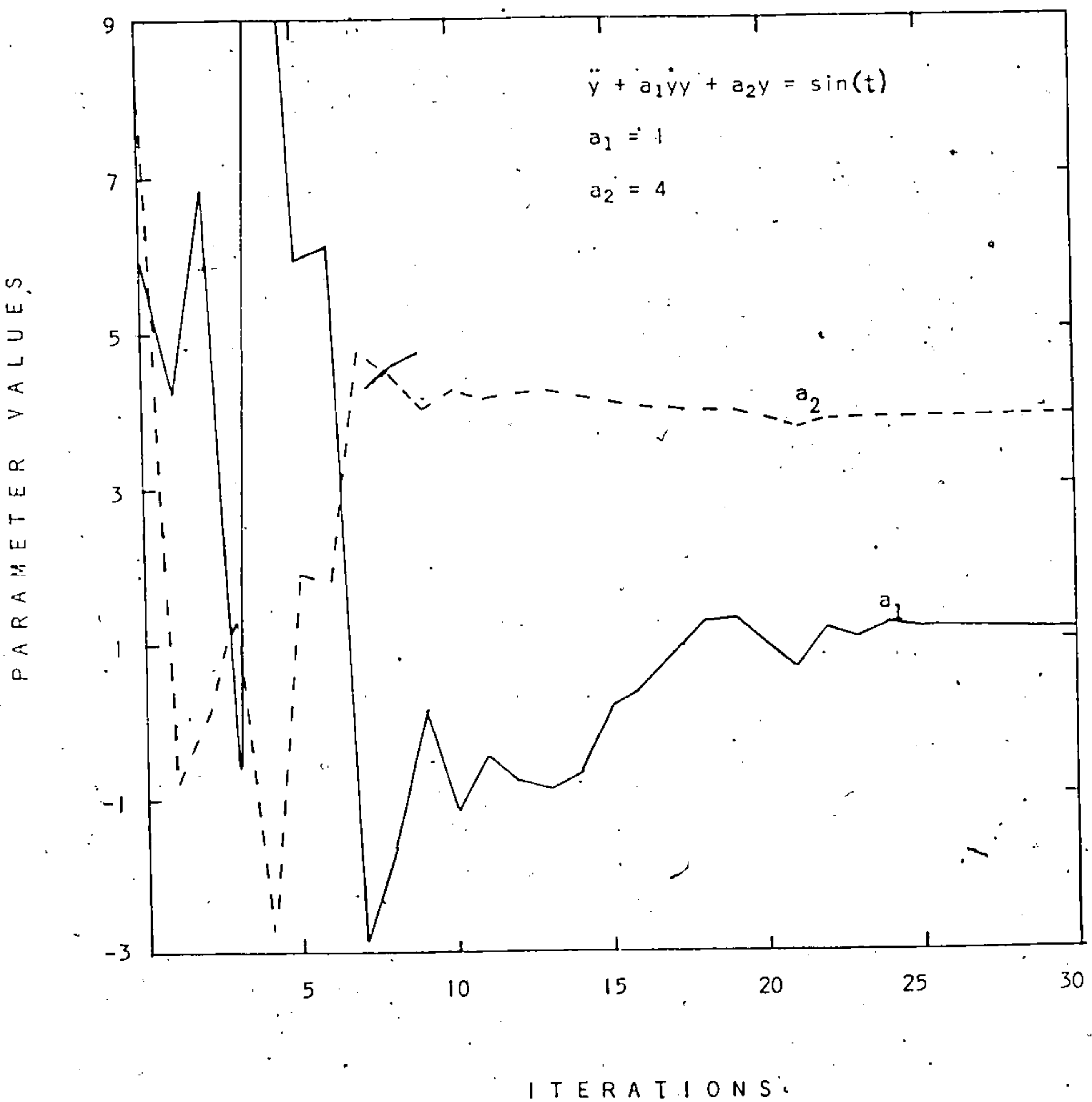

FIGURE 29: Parameter Convergence of System Simulated on EAI 580 Analog Computer

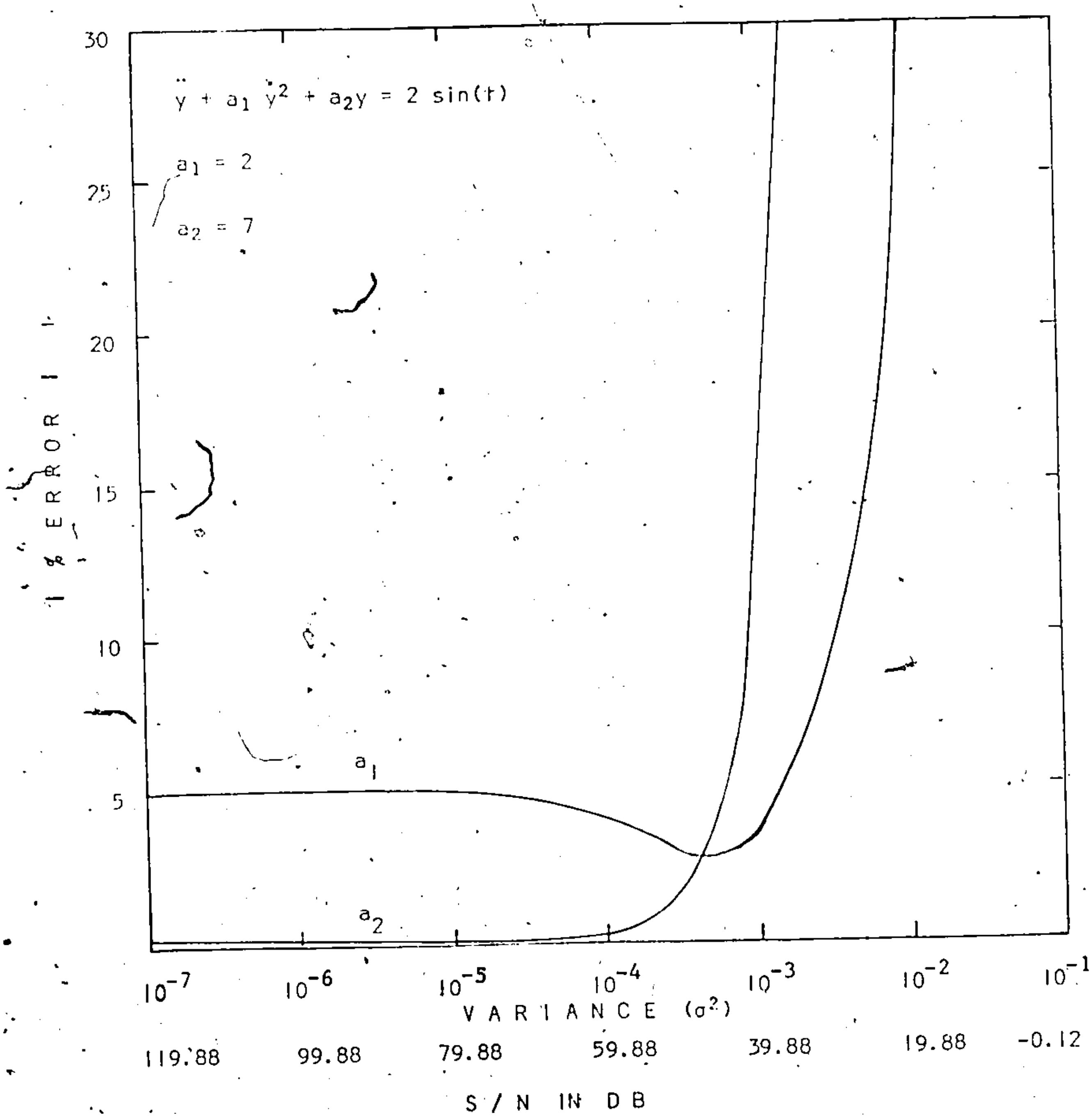

FIGURE 30: Identification After Noise Added to the Analog Simulated System

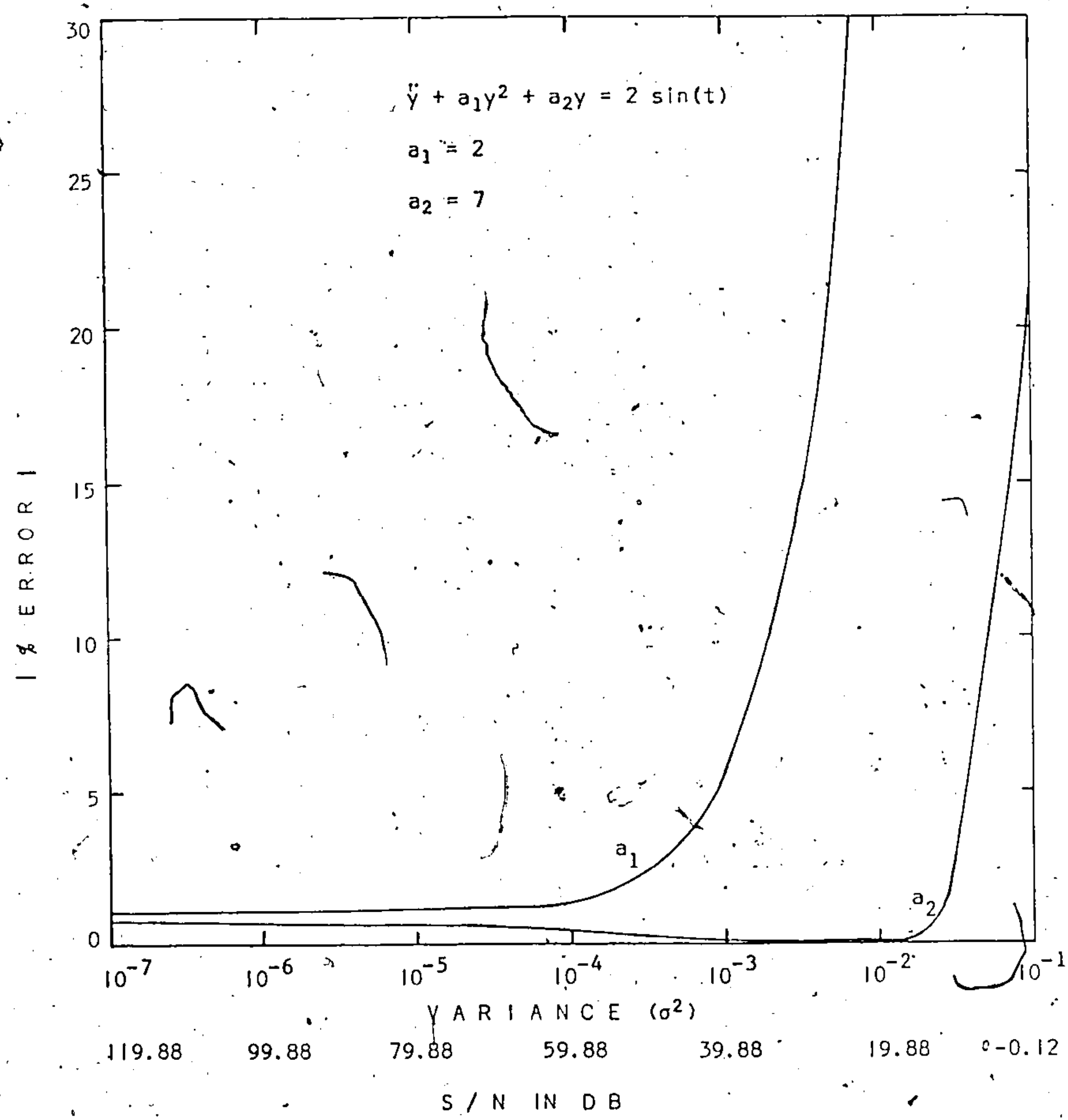

FIGURE 31: Identification of Physical System With Additive Noise by State Variable Approach

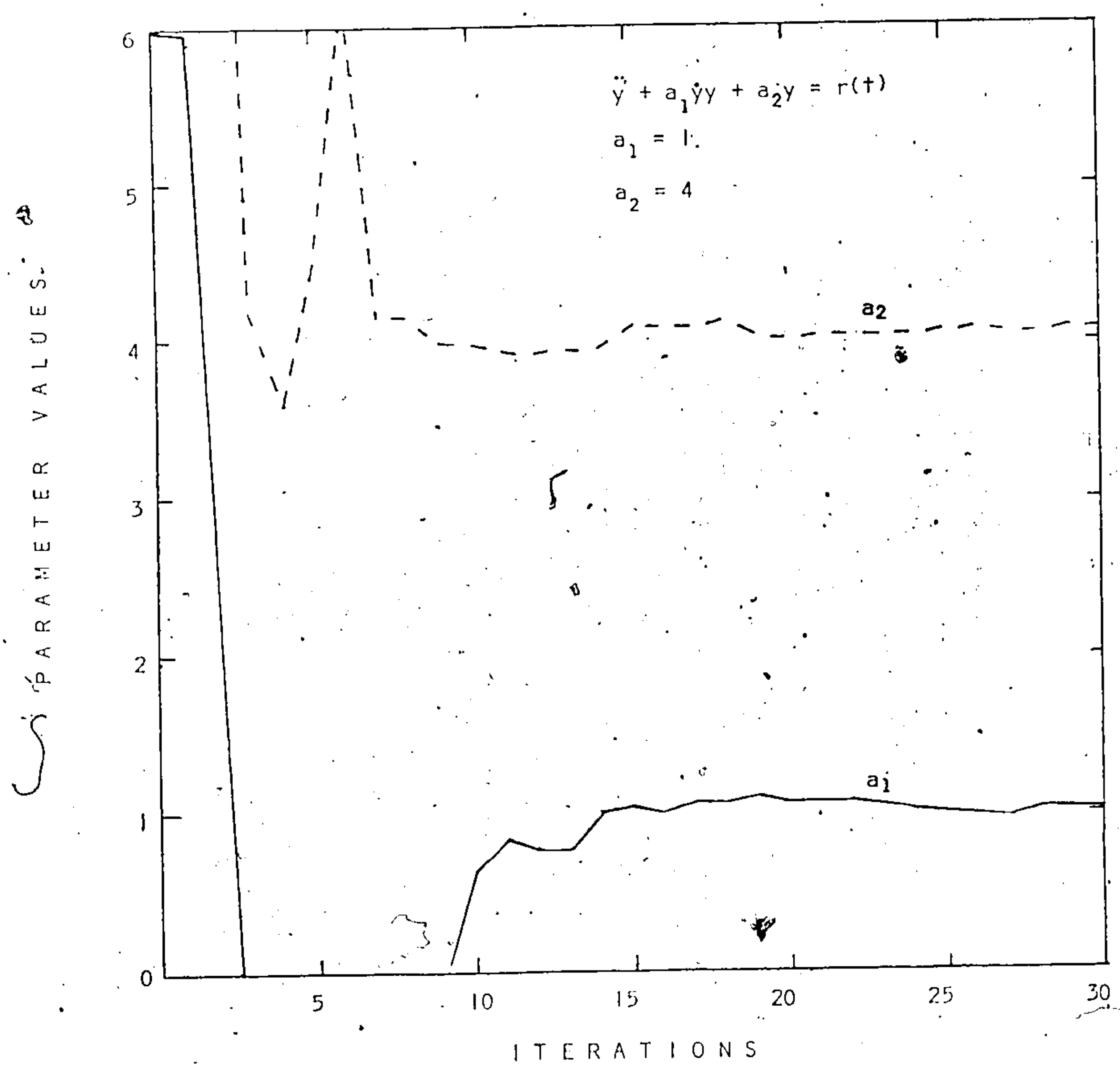

FIGURE 32: Convergence of Parameters of System Simulated on Analog Computer and Excited by Rectangular Input

CHAPTER 6

SUMMARY AND CONCLUSION

The objective of this work was to develop an efficient and simple on-line identification technique for linear time varying and a class of nonlinear systems by interpolating the system output with cubic splines. As the discrete time output measured in practice contains noise, a major requirement was that the developed technique must work with knowledge of this approximate data.

First Balatoni's [6] algorithm for linear stationary lumped parameter has been extended to time varying systems, for parameter variation of unknown nature, by means of on-line recursive relationship. The procedure is capable of tracking time varying unknown parameters associated with a linear dynamical system. As no apriori information about parameter variation was assumed to be known, the parameter variation was treated as random in nature. The effect of additive noise (assumed to be zero mean, wideband) on parameter tracking was also investigated. The technique needs very little apriori information about the noise level in the output.

An on-line technique for identification of a class of nonlinear systems has been developed. The technique utilizes the derivative generation by interpolating the output with cubic splines and solving the resultant linear algebraic equation by recursive least squares procedure given by Lee [18], and can be carried out to any unknown end points. The technique has been shown to be accurate and fast. The technique has been extended for identification of higher order nonlinear systems using the spline on spline technique of derivative generation. Identification of systems described by a set of first order differential equations having multiple inputs and outputs is also discussed. Using state representation of higher order nonlinear system, the application of 'spline on spline' derivative generation for estimating the system states, in conjunction with the recursive least squares procedure, has been shown to be capable of identifying the unknown process parameters. Effect of measurement noise in the identification procedure was considered and the identification was shown to be statistically consistent.

The effect of varying the mesh size in the identification is also considered and guidelines established for finding optimum sampling rates of output data for accurate parameter estimates. The spline technique derived was compared with central difference method and is shown to have superior convergence and accuracy; this is felt to be a direct result of the derivative smoothing properties of cubic splines.

To show the practical feasibility of the techniques developed, a few systems were implemented on an analog computer, sampled by an approximately linear A/D converter and the resulting identification was satisfactory, showing the practical nature of the algorithms.

The main drawback of the said techniques is a definite bias when the output is contaminated with high levels of noise. This is because the recursive least squares technique used introduces a bias in the estimates. It has been shown by Lee [18], that the bias in estimate $\hat{x}_k$ equals

$$x_K - \hat{x}_K = \overline{(A_k' A_k)^{-1} A_k' V_k}$$

where A_k is the coefficient matrix of the system of equations and V_k is a k dimensional noise vector with zero mean and covariance $R = \sigma^2 I$.

The assumption, that A_k and V_k are uncorrelated makes the estimate $\hat{X}_k$ unbiased. However, practically if we insisted on updating the estimate $\hat{X}_k$ only every sample or every few samples, the matrix A will definitely be correlated with V_k and the bias will be finite, and non-zero. When the noise level is high, the bias can be significant. The evaluation of this bias is not a simple task. However, an off-line least square estimate tolerates more noise, since there is no build up of computational error. So when on-line identification is not necessary, a more accurate parameter estimate can be obtained by using off-line least squares estimation, such as Rosenbrocks [28] method of hill climbing.

The major conclusions of this research can be summarized in the following:

(1) The application of cubic splines to on-line identification of linear time varying and nonlinear systems has been shown to be feasible.

(2) It has been shown that the assumption of random variation of the parameters yields the most satisfactory results.

(3) The on-line identification by cubic splines, has been extended to multiple-input-output systems and is seen to be effective.

(4) When on-line identification is not required, the off-line spline identification of systems using state variable approach has been shown to yield more accurate estimates.

(5) The effect of additive noise on parameter estimation is investigated. Spline identification is seen to yield acceptable estimates down to 60 and 40 db of 'signal to noise ratio' for time varying linear and nonlinear systems respectively.

(6) The presence of bias in parameter estimation at higher levels of noise is due to the recursive least square algorithm and not due to the spline approximation. The accuracy of the spline identification to a large extent depends upon the existing stochastic optimal estimators; hence any improvement in the estimation procedure will yield more accurate results.

(7) The use of cubic splines in the identification of systems is promising and it is felt that spline identification can be successfully applied in practice.

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